Lecture Test 2 (1 hour)

Name: **First name:** **Class:**

N.B.: the exam has **4 pages** and 4 exercises.

Exercise 1: probabilities (4 points)

Let $p \in]0,1[$ and X a random variable which is geometric-distributed with parameter p: $X \rightsquigarrow \text{Geom}(p)$.

1. Express the distribution of X.

2. Find its generating function $G_X(t)$. First, express it as a power series, then with the usual functions. Justify.

3. Compute the expectation and the variance of X.

Exercise 2: family of vectors (6 points)

In the vector space $E = \mathcal{M}_2(\mathbb{R})$, consider the family $\mathcal{F} = \left(A = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}, C = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}, D = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \right)$.

1. Write the definition, in this context and with quantifiers, of "$\mathcal{F}$ is linearly independent".

2. Is this family $\mathcal{F}$ linearly independent? Justify.

3. Write the definition, in this context and with quantifiers, of "$\mathcal{F}$ is a spanning family of E".

4. Is $\mathcal{F}$ a spanning family of E? Justify.

5. Is $\mathcal{F}$ a basis of E? If it is, find the coordinates in this basis of $U = \begin{pmatrix} -1 & 1 \\ -3 & 2 \end{pmatrix}$. If not, find dim(Span $\mathcal{F}$). Don't write your computations, but justify your answer.

Exercise 3: linear maps (6 points)

Let E and F be two vector spaces over $\mathbb{R}$ and $f \in \mathcal{L}(E, F)$

1. Write the mathematical definition (with the quantifiers) of $\text{Ker}(f)$ and $\text{Im}(f)$.

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2. Write a necessary and sufficient condition on $\text{Ker}(f)$ and/or $\text{Im}(f)$ for the property "f surjective".

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3. Write and prove a necessary and sufficient condition on $\text{Ker}(f)$ and/or $\text{Im}(f)$ for the property "f injective". Take care to prove this condition is necessary and sufficient.

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4. Find an example of a linear map $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ which is injective and such that $(1, -1, 1) \in \text{Im}(f)$. You don't have to prove that these properties are satisfied.

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Exercise 4: determinant of a square matrix (4 points)

Consider the matrix $A = \begin{pmatrix} 1 & 2 & -1 \\ -2 & 2 & 3 \\ 2 & 1 & -1 \end{pmatrix}$.

1. For each of the following properties, say whether it is true or false. No need to justify.

   (a) For all $B \in \mathcal{M}_3(\mathbb{R})$, $\det(A+B) = \det(A)+\det(B)$. ....

   (b) For all $B \in \mathcal{M}_3(\mathbb{R})$, $\det(AB) = \det(A)\det(B)$. ....

   (c) For all $\lambda \in \mathbb{R}$, $\det(\lambda A) = \lambda \det(A)$. ....

   (d) $\det(A) = \det\left(\begin{pmatrix} 1-1 & 2 & -1 \\ -2+3 & 2 & 3 \\ 2-1 & 1 & -1 \end{pmatrix}\right)$. ....

   (e) $\det(A) = \det\left(\begin{pmatrix} 0 & 2 & -1 \\ 1-1 & 2-1 & 3+1 \\ 1 & 1 & -1 \end{pmatrix}\right)$. ....

2. Compute the determinant of A.

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3. Is the matrix A invertible? Justify.

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