Correction of the midterm exam S3

Exercise 1 (6 points)

1. Find the nature of the series of general term $u_n = \ln\left(\cos\left(\frac{1}{n}\right)\right)$. Justify accurately.

   $u_n = \ln\left(1 - \frac{1}{2n^2} + o\left(\frac{1}{n^2}\right)\right) = -\frac{1}{2n^2} + o\left(\frac{1}{n^2}\right) \sim -\frac{1}{2n^2}$.

   Since $-\frac{1}{2n^2} < 0$ has a constant sign, $\sum u_n$ has the same nature as $-\frac{1}{2}\sum \frac{1}{n^2}$.

   Since $\sum \frac{1}{n^2}$ converges (Riemann), $\sum u_n$ converges too.

2. Find the nature of the series of general term $u_n = \frac{(n!)^2}{(3n)!}$. Justify accurately.

   The sequence (u_n) is strictly positive.

   Furthermore, for all $n \in \mathbb{N}$, $\frac{u_{n+1}}{u_n} = \frac{(n+1)^2}{(3n+3)(3n+2)(3n+1)} \sim \frac{n^2}{27n^3} = \frac{1}{27n}$.

   Hence, $\lim\limits_{n\to+\infty} \frac{u_{n+1}}{u_n} = \lim\limits_{n\to+\infty} \frac{1}{27n} = 0$.

   Since $0 < 1$, $\sum u_n$ converges using the ratio test theorem.

3. Find the nature of the series of general term $u_n = \frac{(-1)^n}{n\ln(n)}$. Justify accurately.

   The sequence (u_n) is alternating. Furthermore, $(|u_n|) = \left(\frac{1}{n\ln(n)}\right)$ is decreasing and converges to 0.

   Finally, $\sum u_n$ converges using Leibniz's theorem.

Exercise 2 (6 points)

Consider the series of general term $u_n = \frac{(-1)^n}{\sqrt{n} + (-1)^n}$.

1. Find $a \in \mathbb{R}$ such that $u_n = \frac{(-1)^n}{\sqrt{n}} + \frac{a}{n} + o\left(\frac{1}{n}\right)$.

   Note that u_n can be written

   $$u_n = \frac{(-1)^n}{\sqrt{n}} \times \frac{1}{1 + \frac{(-1)^n}{\sqrt{n}}}$$

   Using the following TE in 0: $\frac{1}{1+x} = 1 - x + o(x)$, we get

   $$u_n = \frac{(-1)^n}{\sqrt{n}} \times \left(1 - \frac{(-1)^n}{\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right)\right) = \frac{(-1)^n}{\sqrt{n}} - \frac{1}{n} + o\left(\frac{1}{n}\right)$$

   using $-(-1)^{2n} = -1$. Thus,, $a = -1$.

2. Find the nature of $\sum u_n$.

   Let $v_n = \frac{(-1)^n}{\sqrt{n}}$ and $w_n = -\frac{1}{n} + o\left(\frac{1}{n}\right)$.

$\sum v_n$ converges using Leibniz's theorem. Indeed, (v_n) is alternating and $(|v_n|) = \left(\frac{1}{\sqrt{n}}\right)$ is decreasing and converges to 0.

$w_n \sim -\frac{1}{n}$. Note that $-\frac{1}{n} < 0$ has a constant sign. Hence, $\sum w_n$ as he same nature as $\sum \frac{-1}{n}$, which diverges. Thus, $\sum w_n$ diverges.

Finally, $\sum u_n$ diverges because it is the sum of a convergent series and a divergent series.

3. Show that $u_n \sim \frac{(-1)^n}{\sqrt{n}}$

   We have to show that $u_n = \frac{(-1)^n}{\sqrt{n}} + o\left(\frac{(-1)^n}{\sqrt{n}}\right)$.

   Since $u_n = \frac{(-1)^n}{\sqrt{n}} - \frac{1}{n} + o\left(\frac{1}{n}\right)$, we have to show that $-\frac{1}{n} + o\left(\frac{1}{n}\right) = o\left(\frac{(-1)^n}{\sqrt{n}}\right)$.

   But
$$\frac{-\frac{1}{n} + o\left(\frac{1}{n}\right)}{\frac{(-1)^n}{\sqrt{n}}} = (-1)^n\left(\frac{-1}{\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right)\right) \xrightarrow[n\to+\infty]{} 0$$

   Hence, $u_n = \frac{(-1)^n}{\sqrt{n}} + o\left(\frac{(-1)^n}{\sqrt{n}}\right)$, that is, $u_n \sim \frac{(-1)^n}{\sqrt{n}}$.

4. Do the series $\sum u_n$ and $\sum \frac{(-1)^n}{\sqrt{n}}$ have the same nature? Explain why.

   The series don't have the same nature: $\sum u_n$ diverges while $\sum \frac{(-1)^n}{\sqrt{n}}$ converges. Comparison methods are not valid here, because (u_n) has no constant sin.

Exercise 3 (7 points)

Let $a \in]0, \pi[$ and consider the sequence (u_n) defined for all $n \in \mathbb{N}^*$ by

$$u_n = n! \times \prod_{k=1}^{n} \sin\left(\frac{a}{k}\right) = n! \times \left(\sin\left(\frac{a}{1}\right)\sin\left(\frac{a}{2}\right)\cdots\sin\left(\frac{a}{n}\right)\right)$$

Accept without proof that (u_n) is strictly positive. The purpose of the exercise is to study the nature of $\sum u_n$ depending a.

1. Assume in this question that $a \neq 1$. Using the ratio test (d'Alembert test), discuss the nature of $\sum u_n$ depending on a.

   $\frac{u_{n+1}}{u_n} = (n+1)\sin\left(\frac{a}{n+1}\right)$. Furthermore, as x approaches 0, $\sin(x) \sim x$. Hence, as n approaches $+\infty$,
$$\frac{u_{n+1}}{u_n} \sim (n+1) \times \left(\frac{a}{n+1}\right) \sim a \Longrightarrow \lim_{n\to+\infty} \frac{u_{n+1}}{u_n} = a$$

   Finally, using the ratio test, $\sum u_n$ converges if $a < 1$ and diverges if $a > 1$.

2. Assume in this question that $a = 1$. Consider the series $\sum \ln\left(n \sin\left(\frac{1}{n}\right)\right)$ and the sequence (S_n) of its partial sums.

   (a) Show that for all $n \in \mathbb{N}^*$, $S_n = \ln(u_n)$.

   Keep in mind that $n! = \prod_{k=1}^{n} k$. Thus,
$$\ln(u_n) = \ln\left(\prod_{k=1}^{n} k \sin\left(\frac{a}{k}\right)\right) = \sum_{k=1}^{n} \ln\left(k \sin\left(\frac{1}{k}\right)\right) = S_n$$

(b) Study the nature of $\sum \ln\left(n \sin\left(\frac{1}{n}\right)\right)$.

$$\ln\left(n \sin\left(\frac{1}{n}\right)\right) = \ln\left(n \times \left(\frac{1}{n} - \frac{1}{6n^3} + o\left(\frac{1}{n^3}\right)\right)\right) = \ln\left(1 - \frac{1}{6n^2} + o\left(\frac{1}{n^2}\right)\right)$$

Hence,

$$\ln\left(n \sin\left(\frac{1}{n}\right)\right) = -\frac{1}{6n^2} + o\left(\frac{1}{n^2}\right) \sim -\frac{1}{6n^2}$$

Furthermore, $-\frac{1}{6n^2} < 0$ has a constant sign. Hence, $\sum \ln\left(n \sin\left(\frac{1}{n}\right)\right)$ has the same nature as $\sum -\frac{1}{6n^2}$ which converges (Riemann series).

Finally, $\sum \ln\left(n \sin\left(\frac{1}{n}\right)\right)$ converges.

(c) What can you deduce about the sequence (u_n)?

The sequence (S_n) converges according to previous question. Let ℓ denote its limit. Then

$$u_n = e^{S_n} \Longrightarrow u_n \xrightarrow[n\to+\infty]{} e^{\ell}$$

The sequence (u_n) is hence convergent.

(d) Is the series $\sum u_n$ convergent?

The sequence (u_n) converges to e^{ℓ} which is non-zero (because the exponential function does not evaluate to 0). Thus, (u_n) does not converge to 0 and the series $\sum u_n$ diverges.

Exercise 4: some lecture questions and a theorem's proof (5.5 points)

Let (u_n) and (v_n) be two strictly positive sequences.

1. Assume in this question that $(u_n) \leqslant (v_n)$ above a certain rank, that is, there exists $n_0 \in \mathbb{N}$ such that

$$\forall n \in \mathbb{N}, \quad n \geqslant n_0 \Longrightarrow u_n \leqslant v_n$$

In each of the expressions below, replace the dotted lines with one of the symbols $\Longrightarrow$, $\Longleftarrow$ or $\Longleftrightarrow$:

(a) $\sum u_n$ converges $\Longleftarrow$ $\sum v_n$ converges

(b) $\sum u_n$ diverges $\Longrightarrow$ $\sum v_n$ diverges

2. Assume in this question that, as n approaches $+\infty$, $u_n \sim v_n$.

(a) What can you say about the series $\sum u_n$ and $\sum v_n$?

The series have the same nature.

(b) Prove this property. **You will accept without proof the properties of question 1.**

Assume that $(u_n) \sim (v_n)$. Then there exists a sequence (ε_n) such that

$$\forall n \in \mathbb{N}, u_n = v_n \times (1 + \varepsilon_n) \qquad \text{and} \qquad \varepsilon_n \xrightarrow[n\to+\infty]{} 0$$

Since (ε_n) converges to 0, it remains between $-\frac{1}{2}$ and $+\frac{1}{2}$ above a certain rank: there exists $n_0 \in \mathbb{N}$ such that

$$\begin{aligned} \forall n \in \mathbb{N},\ n \geqslant n_0 \quad &\Longrightarrow \quad -\tfrac{1}{2} \leqslant \varepsilon_n \leqslant \tfrac{1}{2} \\ &\Longrightarrow \quad \tfrac{1}{2} \leqslant 1 + \varepsilon_n \leqslant \tfrac{3}{2} \\ &\Longrightarrow \quad \tfrac{1}{2} v_n \leqslant u_n \leqslant \tfrac{3}{2} v_n \end{aligned}$$

If $\sum u_n$ converges then, using property 1.a and the relation $\frac{1}{2}v_n \leqslant u_n$, we know that $\sum \frac{1}{2}v_n$ converges. Thus, $\sum v_n$ converges.

If $\sum u_n$ diverges then, using property 1.b and the relation $u_n \leqslant \frac{3}{2}v_n$, we know that $\sum \frac{3}{2}v_n$ diverges. Thus, $\sum v_n$ diverges.

Exercise 5: probabilities (6.5 points)

A student is doing an exam made of MCQ questions. The exam has 20 questions, each question counts for 1 point. The total mark at the exam is hence a mark out of 20. The questions have neither negative nor intermediate points: at each question, the mark is 0 or 1, no other values are possible.
The student did not prepare his exam and decides to answer randomly. His choices are independent and, at each question, he has a probability $p \in]0,1[$ of choosing the right answer. (same value of p at each question)

1. For all $k \in [\![1,20]\!]$, consider the random variable $X_k =$ "Mark of the student at question k".
   (a) Let $k \in [\![1,20]\!]$. What is the distribution of X_k?

$$X_k(\Omega) = \{0,1\}, \qquad P(X_k{=}0) = 1-p \quad \text{et} \quad P(X_k{=}1) = p$$

   Thus, $X_k \rightsquigarrow \text{Bernoulli}(p)$.

   (b) Find the generating function G_{X_k} of variable X_k.

   $G_{X_k}(t) = (1-p) + pt$

   (c) Using G_{X_k}, compute the expectation and the variance of X_k.

   For all $t \in \mathbb{R}$, $G_{X_k}{}'(t) = p \Longrightarrow \mathrm{E}(X_k) = G_{X_k}{}'(1) = p$

   Furthermore, for all $t \in \mathbb{R}$,

$$G_{X_k}{}''(t) = 0 \Longrightarrow \mathrm{Var}(X_k) = G_{X_k}{}''(1) + \mathrm{E}(X_k) - \mathrm{E}^2(X_k) = 0 + p - p^2 = p(1-p)$$

2. Consider the random variable $Y =$ "Total mark of the student at the exam".
   (a) Find the generating function of variable Y. Justify your answer.

   $Y = X_1 + X_2 + \cdots + X_{20}$. Since he variables X_k are independent, we get:

$$\forall t \in \mathbb{R},\ G_Y(t) = G_{X_1}(t) \times \cdots \times G_{X_{20}}(t) = \big((1-p) + pt\big)^{20}$$

   (b) Find the distribution of Y.

   Let us expand $G_Y(t)$ using the binomial formula: for all $t \in \mathbb{R}$,

$$G_Y(t) = \sum_{k=0}^{20} \binom{20}{k} (pt)^k (1-p)^{20-k} = \sum_{k=0}^{20} \binom{20}{k} p^k (1-p)^{20-k}\, t^k$$

   Thus,

$$Y(\Omega) = [\![0,20]\!] \qquad \text{and} \qquad \forall k \in [\![0,20]\!],\ P(Y{=}k) = \binom{20}{k} p^k (1-p)^{20-k}$$

   (c) Compute the expectation and the variance of Y.

   $Y = X_1 + \cdots + X_{20} \Longrightarrow \mathrm{E}(Y) = \mathrm{E}(X_1) + \cdots + \mathrm{E}(X_{20}) = p + \cdots + p = 20p$
   Furthermore, because the variables X_k are independent,

$$\mathrm{Var}(Y) = \mathrm{Var}(X_1) + \cdots + \mathrm{Var}(X_{20}) = p(1-p) + \cdots + p(1-p) = 20p(1-p)$$

Exercise 6: power series (6 points)

1. Find the radius of convergence R_1 of the power series $\sum \frac{x^n}{n!}$. Justify your answer.

   Ratio theorem for power series: $\dfrac{\left|\frac{1}{(n+1)!}\right|}{\left|\frac{1}{n!}\right|} = \dfrac{n!}{(n+1)!} = \dfrac{1}{n+1} \xrightarrow[n\to+\infty]{} 0.$

   The radius of convergence of the power series is hence $R_1 = +\infty$.

2. Find (do not justify) a simple expression, using the basic functions, of its sum function, defined for all $x \in]-R_1, R_1[$ by

$$f(x) = \sum_{n=0}^{+\infty} \frac{x^n}{n!}$$

This is the exponential series. Its sum function is $f(x) = e^x$

3. Deduce the radius of convergence and a simple expression of the sum function of $\sum \frac{2^n}{n!} x^n$.

We get this series by injecting à $X = 2x$ in the previous series. It converges for all $X \in \mathbb{R}$, that is, for all $x \in \mathbb{R}$.

Its sum is hence $\displaystyle\sum_{n=0}^{+\infty} \frac{(2x)^n}{n!} = e^{2x}$, its radius of convergence is $+\infty$.

4. Find a simple expression of $\displaystyle\sum_{n=3}^{+\infty} \frac{x^n}{(n-3)!}$.

$\displaystyle\sum_{n=3}^{+\infty} \frac{x^n}{(n-3)!} = x^3 \times \sum_{n=3}^{+\infty} \frac{x^{n-3}}{(n-3)!} = x^3 \times \sum_{k=0}^{+\infty} \frac{x^k}{k!}$ by setting $k = n - 3$.

Hence, $\displaystyle\sum_{n=3}^{+\infty} \frac{x^n}{(n-3)!} = x^3\, e^x$.

5. Show that the function $g : x \longmapsto \dfrac{1}{1+2x}$ can be expressed with the form $g(x) = \displaystyle\sum_{n=0}^{+\infty} (-2)^n x^n$.

What is the radius of convergence R_2 of this power series?

The power series $\sum (-2)^n x^n$ can be written $\sum (-2x)^n$. It is a geometric series, it converges if and only if $|-2x| < 1$, that is, if and only if $|x| < \frac{1}{2}$. Thus, $R_2 = \frac{1}{2}$. Furthermore, when the series converges, its sum is

$$\sum_{n=0}^{+\infty} (-2x)^n = \frac{1}{1-(-2x)} = \frac{1}{1+2x}$$

6. Express the function $x \longmapsto \ln(1+2x)$ as a power series and find its radius of convergence.

Let us integrate the power series of previous question: we get

$$\int_0^x \frac{1}{1+2t}\,\mathrm{d}t = \frac{1}{2}\ln(1+2x) = \sum_{n=0}^{+\infty} (-2)^n \frac{x^{n+1}}{n+1}$$

and the radius of convergence is the same as at previous question. It results that

$$\ln(1+2x) = \sum_{n=0}^{+\infty} 2 \times (-2)^n \frac{x^{n+1}}{n+1} = -\sum_{n=0}^{+\infty} (-2)^{n+1} \frac{x^{n+1}}{n+1}$$

The radius of convergence is $\frac{1}{2}$.

7. Express the function $x \longmapsto \dfrac{x^2}{(1+2x)^2}$ as a power series and find its radius of convergence.

Let us differentiate the power series of question 5. We get

$$\frac{-2}{(1+2x)^2} = \sum_{n=1}^{+\infty} (-2)^n n\, x^{n-1} \qquad \text{and the radius of convergence is } R_2 = \frac{1}{2}$$

Hence,

$$\frac{1}{(1+2x)^2} = \sum_{n=1}^{+\infty} (-2)^{n-1} n\, x^{n-1} \Longrightarrow \frac{x^2}{(1+2x)^2} = \sum_{n=1}^{+\infty} (-2)^{n-1} n\, x^{n+1}$$

Exercise 7: infinite probabilities (4 points)

Consider a random variable X admitting a generating function of the form $G_X(t) = a\,e^{2t}$, where $a \in \mathbb{R}$.

1. What is the value of a?

   $G_X(1) = 1 \Longrightarrow a\,e^2 = 1 \Longrightarrow a = e^{-2}$

2. By expressing $G_X(t)$ as a power series, find the distribution of X.

   Let us express $G_X(t)$ as a power series:

$$G_X(t) = e^{-2} \sum_{n=0}^{+\infty} \frac{(2t)^n}{n!} = \sum_{n=0}^{+\infty} e^{-2} \frac{2^n}{n!}\, t^n$$

   Thus,

$$X(\Omega) = \mathbb{N} \qquad \text{and} \qquad \forall n \in \mathbb{N},\ P(X{=}n) = e^{-2} \frac{2^n}{n!}$$

3. Compute the expectation and the variance of X.

   For all $t \in \mathbb{R}$, $\quad G_X{}'(t) = e^{-2} \times 2e^{2t} \quad$ and $\quad G_X{}''(t) = e^{-2} \times 4e^{2t}$

   Thus, $\quad \mathrm{E}(X) = G_X{}'(1) = e^{-2} \times 2e^2 = 2.$

   And $\quad \mathrm{Var}(X) = G_X{}''(1) + \mathrm{E}(X) - \mathrm{E}^2(X) = 4 + 2 - 4 = 2.$