EPITA

Mathematics

Midterm exam S3

November 2022

Duration: 3 hours

Name:

First name:

Class:

MARK:

The marking system is for a mark from 0 to 40. It will be divided by 2, to get a mark from 0 to 20.

Instructions:

- Documents and pocket calculators are not allowed.
- Write your answers on the stapled sheets provided for answering. No other sheet will be corrected.
- Please, do not use lead pencils for answering.

Exercise 1 (6 points)

1. Find the nature of the series whose general term is: $u_n = \dfrac{\sin\left(\frac{1}{n}\right)}{n^2}$. Justify carefully.

2. Find the nature of the series whose general term is: $u_n = \dfrac{n^2 e^{-\sqrt{n}}}{2^{2n}}$. Justify carefully.

3. Find the nature of the series whose general term is: $u_n = (-1)^n \dfrac{n}{e^n}$. Justify carefully.

Exercise 2 (6 points)

Let $a \in \mathbb{R}$ such that $a > 0$ and consider the sequence (u_n) defined for all $n \geqslant 2$ by: $u_n = \dfrac{(-1)^n}{\sqrt{n^a + (-1)^n}}$.

The purpose of the exercise is to study the nature of $\sum u_n$.

1. Find $c \in \mathbb{R}$ such that $u_n = \dfrac{(-1)^n}{n^{a/2}} + \dfrac{c}{n^{3a/2}} + o\left(\dfrac{1}{n^{3a/2}}\right)$.

2. Using the previous question, discuss the nature of $\sum u_n$ depending on the value of a.

Exercise 3 (8 points)

The purpose of this exercise is to study the sequence (u_n) defined for all $n \in \mathbb{N}^*$ by: $u_n = \dfrac{2^n n!}{1 \times 3 \times 5 \times \cdots \times (2n-1)}$.

In that purpose, consider the auxiliary sequence (v_n) defined by: $v_n = \ln(u_n) - \dfrac{1}{2}\ln(n)$.

1. Let $n \in \mathbb{N}^*$. Compute $\dfrac{u_{n+1}}{u_n}$.

2. Deduce that $v_{n+1} - v_n = \dfrac{1}{2}\ln\left(1 + \dfrac{1}{n}\right) - \ln\left(1 + \dfrac{1}{2n}\right)$.

3. Find $a \in \mathbb{R}$ such that $v_{n+1} - v_n \sim \dfrac{a}{n^2}$.

4. Show that (v_n) converges. Let ℓ denote its limit.

5. Using the previous question, show that there exists $k \in \mathbb{R}$ such that $u_n \sim k\sqrt{n}$. Express k as a function of ℓ.

Exercise 4: Leibniz's rule (5 points)

Let (u_n) be a numerical sequence with an alternating sign.

1. Write the statement of the Leibniz's theorem about $\sum u_n$.

2. Prove this theorem.

   N.B.: prove only the convergence of the series $\sum u_n$. It is not required to prove the upper bound of the remainders.

Exercise 5: probabilities (5 points)

Let X_1, X_2 and X_3 be three independent random variables, taking their values in $\{1, 3\}$, such that for all $i \in [\![1, 3]\!]$,

$$P(X_i{=}1) = \frac{1}{3} \qquad \text{and} \qquad P(X_i{=}3) = \frac{2}{3}$$

1. What are the generating functions G_{X_i} of these variables?

2. Consider the random variable $Y = X_1 + X_2 + X_3$. Compute its generating function G_Y and deduce its distribution.

3. Compute the expectation and the variance of Y.

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Exercise 6: power series (10 points)

Our purpose is to find a function f satisfying the following conditions $(C) : \begin{cases} f'' + xf' + f = 0 \\ f(0) = 1 \text{ and } f'(0) = 0 \end{cases}$

In that purpose, assume there exists a power series $\sum a_n x^n$, admitting a radius of convergence $R > 0$, such that:

$$f(x) = \sum_{n=0}^{+\infty} a_n x^n \qquad \text{and} \qquad f \text{ solution of } (C) \text{ on }]-R, R[$$

Note: the differential equation $f'' + xf' + f = 1$ is an order 2 linear equation with non-constant coefficients. Thus, **don't try to use the methods that you have studied at the S2** about order 2 differential equations: the latter methods only work for constant coefficients equations.

1. Express $f(0)$ and $f'(0)$ as functions of the sequence (a_n). What can you deduce about the values of a_0 and a_1?

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2. Define, as functions of (a_n), two sequences (b_n) and (c_n) such that for all $x \in]-R, R[$,

$$xf'(x) = \sum_{n=0}^{+\infty} b_n x^n \qquad \text{and} \qquad f''(x) = \sum_{n=0}^{+\infty} c_n x^n$$

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3. By injecting these expressions of $xf'(x)$ and $f''(x)$ into the equation $f''(x)+xf'(x)+f(x)=0$, write this equation as:

$$\forall x \in]-R,R[, \quad \sum_{n=0}^{+\infty} d_n x^n = 0$$

where the coefficients (d_n) depend on the sequence (a_n).

4. Keep in mind that the condition $\sum_{n=0}^{+\infty} d_n x^n = 0$ implies that all the d_n equal zero. Using this property, show that

$$a_2 = -\frac{1}{2}, \qquad a_3 = 0 \qquad \text{and, as a general rule:} \qquad \forall \in \mathbb{N},\ a_{n+2} = -\frac{a_n}{n+2}$$

5. What is the value of a_n when n is odd?

6. Find the value of a_n when n is even.

   Hint: you can set $n = 2k$ ($k \in \mathbb{N}$). To start with, express a_{2k} as a function of $a_{2(k-1)}$, then as a function of $a_{2(k-2)}$, and so on, until you get an expression of a_{2k} as a function of a_0.

7. Deduce $f(x)$. First, express it as a power series, then find an expression with the usual functions.

8. **(Bonus)** Check that the final expression that you got at previous question is a solution of (C) on the whole set $\mathbb{R}$.