Exercise 1 (6 points)

1. Find the nature of the series whose general term is: $u_n = \dfrac{\sin\left(\frac{1}{n}\right)}{n^2}$. Justify carefully.

   $\sin\left(\frac{1}{n}\right) \sim \frac{1}{n} \Longrightarrow u_n \sim \dfrac{1}{n^3}$.

   Since $\dfrac{1}{n^3} \geqslant 0$, $\sum u_n$ has the nature of $\sum \dfrac{1}{n^3}$ which converges. Hence, $\sum u_n$ converges.

2. Find the nature of the series whose general term is: $u_n = \dfrac{n^2 e^{-\sqrt{n}}}{2^{2n}}$. Justify carefully.

   The sequence (u_n) is positive. Furthermore, $n^2 u_n = \dfrac{n^4}{e^{\sqrt{n}+n\ln(4)}} \xrightarrow[n\to+\infty]{} 0$, that is, $u_n = o\left(\dfrac{1}{n^2}\right)$.

   Since $\sum \dfrac{1}{n^2}$ converges, $\sum u_n$ converges.

   Remark: you can also use the ratio or the root test. You will get $\lim\limits_{n\to+\infty} \dfrac{u_{n+1}}{u_n} = \lim\limits_{n\to+\infty} \sqrt[n]{u_n} = \dfrac{1}{4}$.

3. Find the nature of the series whose general term is: $u_n = (-1)^n \dfrac{n}{e^n}$. Justify carefully.

   $n^2|u_n| = \dfrac{n^3}{e_n} \xrightarrow[n\to+\infty]{} 0$, that is, $|u_n| = o\left(\dfrac{1}{n^2}\right)$.

   Since $\sum \dfrac{1}{n^2}$ converges, $\sum |u_n|$ converges. Finally, $\sum u_n$ converges absolutely, so it converges.

   Remark: you can also use the Leibniz's rule. The sequence $(|u_n|)$ decreases above the rank $n = 1$.

Exercise 2 (6 points)

Let $a \in \mathbb{R}$ such that $a > 0$ and consider the sequence (u_n) defined for all $n \geqslant 2$ by: $u_n = \dfrac{(-1)^n}{\sqrt{n^a + (-1)^n}}$.

The purpose of the exercise is to study the nature of $\sum u_n$.

1. Find $c \in \mathbb{R}$ such that $u_n = \dfrac{(-1)^n}{n^{a/2}} + \dfrac{c}{n^{3a/2}} + o\left(\dfrac{1}{n^{3a/2}}\right)$.

$$
\begin{aligned}
u_n &= \frac{(-1)^n}{\sqrt{n^a}} \times \frac{1}{\sqrt{1 + \frac{(-1)^n}{n^a}}} \\
&= \frac{(-1)^n}{n^{a/2}} \times \left(1 + \frac{(-1)^n}{n^a}\right)^{-1/2} \\
&= \frac{(-1)^n}{n^{a/2}} \times \left(1 - \frac{(-1)^n}{2n^a} + o\left(\frac{1}{n^a}\right)\right) \\
&= \frac{(-1)^n}{n^{a/2}} - \frac{1}{2n^{3a/2}} + o\left(\frac{1}{n^{3a/2}}\right)
\end{aligned}
$$

   Hence, $c = -\dfrac{1}{2}$.

2. Using the previous question, discuss the nature of $\sum u_n$ depending on the value of a.

   Let $v_n = \dfrac{(-1)^n}{n^{a/2}}$ and $w_n = -\dfrac{1}{2n^{3a/2}} + o\left(\dfrac{1}{n^{3a/2}}\right)$

   The series $\sum u_n$ is the sum of the two series $\sum v_n$ and $\sum w_n$.

   - $\sum v_n$ converges according to the Leibniz's rule. Indeed, (v_n) is alternating, $(|v_n|) = \left(\dfrac{1}{n^{a/2}}\right)$ is decreasing and converges to 0.

- $w_n \sim -\dfrac{1}{2n^{3a/2}} < 0$. Since $-\dfrac{1}{2n^{3a/2}}$ has a constant sign, $\sum w_n$ has the same nature as $\sum -\dfrac{1}{2n^{3a/2}}$. The latter series is a Riemann series, it converges iff $a > \frac{2}{3}$.
- As a conclusion:
  If $a > \frac{2}{3}$, $\sum u_n$ is the sum of two convergent series, it hence converges.
  If $a \leqslant \frac{2}{3}$, $\sum u_n$ is the sum of a convergent series and a divergent series. It hence diverges.

Exercise 3 (8 points)

The purpose of this exercise is to study the sequence (u_n) defined for all $n \in \mathbb{N}^*$ by: $u_n = \dfrac{2^n n!}{1 \times 3 \times 5 \times \cdots \times (2n-1)}$.

In that purpose, consider the auxiliary sequence (v_n) defined by: $v_n = \ln(u_n) - \dfrac{1}{2}\ln(n)$.

1. Let $n \in \mathbb{N}^*$. Compute $\dfrac{u_{n+1}}{u_n}$.

$$\frac{u_{n+1}}{u_n} = \frac{2(n+1)}{2(n+1)-1} = \frac{2n+2}{2n+1} = \frac{1+\frac{1}{n}}{1+\frac{1}{2n}}$$

2. Deduce that $v_{n+1} - v_n = \dfrac{1}{2}\ln\left(1+\dfrac{1}{n}\right) - \ln\left(1+\dfrac{1}{2n}\right)$.

$$\begin{aligned}
v_{n+1} - v_n &= \ln(u_{n+1}) - \frac{1}{2}\ln(n+1) - \ln(u_n) + \frac{1}{2}\ln(n) \\
&= \ln\left(\frac{u_{n+1}}{u_n}\right) - \frac{1}{2}\ln\left(\frac{n+1}{n}\right) \\
&= \ln\left(\frac{1+\frac{1}{n}}{1+\frac{1}{2n}}\right) - \frac{1}{2}\ln\left(1+\tfrac{1}{n}\right) \\
&= \ln\left(1+\tfrac{1}{n}\right) - \ln\left(1+\tfrac{1}{2n}\right) - \frac{1}{2}\ln\left(1+\tfrac{1}{n}\right) \\
&= \frac{1}{2}\ln\left(1+\tfrac{1}{n}\right) - \ln\left(1+\tfrac{1}{2n}\right)
\end{aligned}$$

3. Find $a \in \mathbb{R}$ such that $v_{n+1} - v_n \sim \dfrac{a}{n^2}$.

$$\begin{aligned}
v_{n+1} - v_n &= \frac{1}{2}\ln\left(1+\tfrac{1}{n}\right) - \ln\left(1+\tfrac{1}{2n}\right) \\
&= \frac{1}{2}\left(\frac{1}{n} - \frac{1}{2n^2} + o\left(\frac{1}{n^2}\right)\right) - \left(\frac{1}{2n} - \frac{1}{8n^2} + o\left(\frac{1}{n^2}\right)\right) \\
&= \frac{1}{2n} - \frac{1}{4n^2} - \frac{1}{2n} + \frac{1}{8n^2} + o\left(\frac{1}{n^2}\right) \\
&= -\frac{1}{8n^2} + o\left(\frac{1}{n^2}\right) \\
&\sim -\frac{1}{8n^2}
\end{aligned}$$

Thus, $a = -\frac{1}{8}$.

4. Show that (v_n) converges. Let ℓ denote its limit.

$v_{n+1} - v_n \sim -\dfrac{1}{8n^2} < 0$. Since $-\dfrac{1}{8n^2}$ has a constant sign, $\sum(v_{n+1} - v_n)$ has the same nature as $\sum -\dfrac{1}{8n^2}$. And the latter series converges.

Thus, $\sum(v_{n+1} - v_n)$ converges and, using telescopic theorem, (v_n) converges too.

5. Using the previous question, show that there exists $k \in \mathbb{R}$ such that $u_n \sim k\sqrt{n}$. Express k as a function of ℓ.

   To start with, note that $v_n = \ln(u_n) - \frac{1}{2}\ln(n) \Longrightarrow \ln(u_n) = v_n + \frac{1}{2}\ln(n) \Longrightarrow u_n = e^{v_n}\sqrt{n}$.

   We hence get $\frac{u_n}{e^{\ell}\sqrt{n}} = \frac{e^{v_n}}{e^{\ell}} = e^{v_n-\ell} \xrightarrow[n\to+\infty]{} e^0 = 1$

   Finally, $v_n \sim e^{\ell}\sqrt{n}$ and $k = e^{\ell}$.

Exercise 4: Leibniz's rule (5 points)

Let (u_n) be a numerical sequence with an alternating sign.

1. Write the statement of the Leibniz's theorem about $\sum u_n$.

   If the sequence $\big(|u_n|\big)$ is decreasing and converging to 0 then:

   - The series $\sum u_n$ converges.
   - The sequence (R_n) of the series' remainders satisfies to: $\forall n \in \mathbb{N}, |R_n| \leqslant |u_{n+1}|$

2. Prove this theorem.

   N.B.: prove only the convergence of the series $\sum u_n$. It is not required to prove the upper bound of the remainders.

   The sequence (u_n) is alternating. Thus, there exists a positive sequence (a_n) such that

   $$(u_n) = \Big((-1)^n a_n\Big) \qquad \text{or} \qquad (u_n) = \Big(-(-1)^n a_n\Big)$$

   For the proof, we can assume that we are in the first case $(u_n) = \Big((-1)^n a_n\Big)$. If not, just replace (u_n) with $(-u_n)$. The positive sequence (a_n) is in fact the sequence $(|u_n|)$: the theorem's hypotheses state that it decreases and converges to 0.

   Let (S_n) be the partial sums of $\sum u_n$: for all $n \in \mathbb{N}$,

   $$S_n = a_0 - a_1 + a_2 - a_3 + \cdots + (-1)^n a_n$$

   To start with, let us prove that the sequences (S_{2n}) and (S_{2n+1}) are adjacent.

   (a) Monotony of (S_{2n}): this subsequence contains the terms of even ranks. The term following S_{2n} is hence $S_{2(n+1)} = S_{2n+2}$. Thus, for all $n \in \mathbb{N}$:

   $$\left\{\begin{array}{rcl} S_{2n} & = & a_0 - a_1 + a_2 - a_3 + \cdots + a_{2n} \\ S_{2(n+1)} & = & a_0 - a_1 + a_2 - a_3 + \cdots + a_{2n} - a_{2n+1} + a_{2n+2} \\ \hline S_{2(n+1)} - S_{2n} & = & -a_{2n+1} + a_{2n+2} \end{array}\right.$$

   Since (a_n) is decreasing, $-a_{2n+1} + a_{2n+2}$ is negative. The sequence (S_{2n}) is hence decreasing.

   (b) Monotony of (S_{2n+1}): this subsequence contains the terms of odd ranks. The term following S_{2n+1} is hence $S_{2(n+1)+1} = S_{2n+3}$. Thus, for all $n \in \mathbb{N}$:

   $$\left\{\begin{array}{rcl} S_{2n+1} & = & a_0 - a_1 + a_2 - a_3 + \cdots + a_{2n} - a_{2n+1} \\ S_{2(n+1)+1} & = & a_0 - a_1 + a_2 - a_3 + \cdots + a_{2n} - a_{2n+1} + a_{2n+2} - a_{2n+3} \\ \hline S_{2(n+1)+1} - S_{2n+1} & = & a_{2n+2} - a_{2n+3} \end{array}\right.$$

   Since (a_n) is decreasing, $a_{2n+2} - a_{2n+3}$ is positive. The sequence (S_{2n+1}) is hence increasing.

   (c) Study of $S_{2n+1} - S_{2n}$: for all $n \in \mathbb{N}$,

   $$\left\{\begin{array}{rcl} S_{2n} & = & a_0 - a_1 + a_2 - a_3 + \cdots + a_{2n} \\ S_{2n+1} & = & a_0 - a_1 + a_2 - a_3 + \cdots + a_{2n} - a_{2n+1} \\ \hline S_{2n+1} - S_{2n} & = & -a_{2n+1} \end{array}\right.$$

   Since (a_n) converges to 0, $(S_{2n+1} - S_{2n})$ converges to 0 too.

We hence proved that the sequences (S_{2n}) and (S_{2n+1}) are adjacent. From this, we know that they both converge and admit an identical limit ℓ. Then we get:

$$\left.\begin{array}{ccc} S_{2n} & \xrightarrow[n\to+\infty]{} & \ell \\ S_{2n+1} & \xrightarrow[n\to+\infty]{} & \ell \end{array}\right\} \Longrightarrow S_n \xrightarrow[n\to+\infty]{} \ell$$

This proves that (S_n) converges, that is, $\sum u_n$ converges.

Exercise 5: probabilities (5 points)

Let X_1, X_2 and X_3 be three independent random variables, taking their values in $\{1,3\}$, such that for all $i \in [\![1,3]\!]$,

$$P(X_i{=}1) = \frac{1}{3} \qquad \text{and} \qquad P(X_i{=}3) = \frac{2}{3}$$

1. What are the generating functions G_{X_i} of these variables?

   $G_{X_1}(t) = G_{X_2}(t) = G_{X_3}(t) = \dfrac{t+2t^3}{3}$

2. Consider the random variable $Y = X_1 + X_2 + X_3$. Compute its generating function G_Y and deduce its distribution.

   Since the X_i are independent, $G_Y(t) = G_{X_1}(t) \times G_{X_2}(t) \times G_{X_3}(t) = \dfrac{(t+2t^3)^3}{3^3} = \dfrac{t^3(1+2t^2)^3}{27}$

   Thus, $\quad G_Y(t) = \dfrac{t^3(1+6t^2+12t^4+8t^6)}{27} = \dfrac{t^3+6t^5+12t^7+8t^9}{27}$.

   The distribution of Y is given by:

   $$Y(\Omega) = \{3,5,7,9\} \quad \text{with} \quad P(Y{=}3) = \frac{1}{27},\ P(Y{=}5) = \frac{6}{27} = \frac{2}{9},\ P(Y{=}7) = \frac{12}{27} = \frac{4}{9} \text{ and } P(Y{=}9) = \frac{8}{27}$$

3. Compute the expectation and the variance of Y.

   On the one hand, $\mathrm{E}(Y) = \mathrm{E}(X_1) + \mathrm{E}(X_2) + \mathrm{E}(X_3)$. On the other hand, since the X_i are independent, $\mathrm{Var}(Y) = \mathrm{Var}(X_1) + \mathrm{Var}(X_2) + \mathrm{Var}(X_3)$.

   - $G_{X_i}(t) = \dfrac{t+2t^3}{3} \Longrightarrow G'_{X_i}(t) = \dfrac{1+6t^2}{3} \Longrightarrow \mathrm{E}(X_i) = G'_{X_i}(1) = \dfrac{7}{3}$.

     Hence, $\mathrm{E}(Y) = \mathrm{E}(X_1) + \mathrm{E}(X_2) + \mathrm{E}(X_3) = 7$.
   - $G''_{X_i}(t) = \dfrac{12t}{3} = 4t \Longrightarrow \mathrm{Var}(X_i) = G''_{X_i}(1) + \mathrm{E}(X_i) - \mathrm{E}^2(X_i) = 4 + \dfrac{7}{3} - \dfrac{49}{9} = \dfrac{8}{9}$.

     Hence, $\mathrm{Var}(Y) = \mathrm{Var}(X_1) + \mathrm{Var}(X_2) + \mathrm{Var}(X_3) = \dfrac{8}{3}$.

Exercise 6: power series (10 points)

Our purpose is to find a function f satisfying the following conditions (C) : $\begin{cases} f'' + xf' + f = 0 \\ f(0) = 1 \text{ and } f'(0) = 0 \end{cases}$

In that purpose, assume there exists a power series $\sum a_n x^n$, admitting a radius of convergence $R > 0$, such that:

$$f(x) = \sum_{n=0}^{+\infty} a_n x^n \qquad \text{and} \qquad f \text{ solution of } (C) \text{ on }]-R, R[$$

Note: the differential equation $f'' + xf' + f = 1$ is an order 2 linear equation with non-constant coefficients. Thus, **don't try to use the methods that you have studied at the S2** about order 2 differential equations: the latter methods only work for constant coefficients equations.

1. Express $f(0)$ and $f'(0)$ as functions of the sequence (a_n). What can you deduce about the values of a_0 and a_1?

$f(x) = \sum_{n=0}^{+\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots \Longrightarrow f(0) = a_0$

$f'(x) = \sum_{n=0}^{+\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots \Longrightarrow f'(0) = a_1$

Using the conditions $f(0) = 1$ and $f'(0) = 0$, we get: $a_0 = 1$ and $a_1 = 0$.

2. Define, as functions of (a_n), two sequences (b_n) and (c_n) such that for all $x \in]-R, R[$,

$$xf'(x) = \sum_{n=0}^{+\infty} b_n x^n \qquad \text{and} \qquad f''(x) = \sum_{n=0}^{+\infty} c_n x^n$$

$f'(x) = \sum_{n=0}^{+\infty} n a_n x^{n-1} \Longrightarrow xf'(x) = \sum_{n=0}^{+\infty} n a_n x^n$. Thus, $(b_n) = (na_n)$.

$f''(x) = \sum_{n=0}^{+\infty} n(n-1) a_n x^{n-2} = (2\times 1)a_2 + (3\times 2)a_3 x + (4\times 3)x^2 + \cdots = \sum_{n=0}^{+\infty} (n+2)(n+1)a_{n+2} x^n$.

Thus, $(c_n) = \big((n+2)(n+1)a_{n+2}\big)$.

3. By injecting these expressions of $xf'(x)$ and $f''(x)$ into the equation $f''(x) + xf'(x) + f(x) = 0$, write this equation as:

$$\forall x \in]-R, R[, \quad \sum_{n=0}^{+\infty} d_n x^n = 0$$

where the coefficients (d_n) depend on the sequence (a_n).

$$\begin{aligned} \forall x \in]-R, R[, f''(x) + xf'(x) + f(x) = 0 \quad &\Longrightarrow \quad \forall x \in]-R, R[, \sum_{n=0}^{+\infty} c_n x^n + \sum_{n=0}^{+\infty} b_n x^n + \sum_{n=0}^{+\infty} a_n x^n = 0 \\ &\Longrightarrow \quad \forall x \in]-R, R[, \sum_{n=0}^{+\infty} (c_n + b_n + a_n) x^n = 0 \end{aligned}$$

Thus, for all $n \in \mathbb{N}$, $d_n = (n+2)(n+1)a_{n+2} + na_n + a_n = (n+1)\big((n+2)a_{n+2} + a_n\big)$

4. Keep in mind that the condition $\sum_{n=0}^{+\infty} d_n x^n = 0$ implies that all the d_n equal zero. Using this property, show that

$$a_2 = -\frac{1}{2}, \qquad a_3 = 0 \qquad \text{and, as a general rule:} \qquad \forall \in \mathbb{N},\ a_{n+2} = -\frac{a_n}{n+2}$$

Since all the coefficients d_n are zeros, we deduce that for all $n \in \mathbb{N}$:

$$(n+1)\big((n+2)a_{n+2} + a_n\big) = 0 \Longrightarrow (n+2)a_{n+2} + a_n = 0 \Longrightarrow a_{n+2} = -\frac{a_n}{n+2}$$

For example, since $a_0 = 1$ and $a_1 = 0$, we get $a_2 = -\frac{a_0}{2} = -\frac{1}{2}$ and $a_3 = -\frac{a_1}{3} = 0$.

5. What is the value of a_n when n is odd?

$a_1 = 0, \quad a_3 = -\frac{a_1}{3} = 0, \quad a_5 = -\frac{a_3}{5} = 0, \quad \cdots$

Finally, all the coefficients a_n are zeros when n is odd.

6. Find the value of a_n when n is even.

   Hint: you can set $n = 2k$ ($k \in \mathbb{N}$). To start with, express a_{2k} as a function of $a_{2(k-1)}$, then as a function of $a_{2(k-2)}$, and so on, until you get an expression of a_{2k} as a function of a_0.

   $$a_{2k} = -\frac{a_{2(k-1)}}{2k} = \left(-\frac{1}{2k}\right) \times a_{2(k-1)} = \left(-\frac{1}{2k}\right) \times \left(-\frac{1}{2(k-1)}\right) \times a_{2(k-2)} = \cdots$$

   If we go on, we get: $a_{2k} = \left(-\frac{1}{2k}\right) \times \left(-\frac{1}{2(k-1)}\right) \times \cdots \times \left(-\frac{1}{2}\right) a_0$

   Since $a_0 = 1$, we get $a_{2k} = \dfrac{(-1)^k}{2^k\, k!} = \dfrac{\left(-\frac{1}{2}\right)^k}{k!}$.

7. Deduce $f(x)$. First, express it as a power series, then find an expression with the usual functions.

   Remind that $\quad f(x) = \displaystyle\sum_{n=0}^{+\infty} a_n x^n.$

   But in this sum, only the terms of even ranks are non-zeros. Thus, we can write it as:

   $$f(x) = \sum_{k=0}^{+\infty} a_{2k} x^{2k} = \sum_{k=0}^{+\infty} \frac{\left(-\frac{1}{2}\right)^k}{k!} x^{2k}$$

   Note that for all $k \in \mathbb{N}$, $\dfrac{\left(-\frac{1}{2}\right)^k}{k!} x^{2k} = \dfrac{\left(-\frac{x^2}{2}\right)^k}{k!}$. Thus, we get:

   $$f(x) = \sum_{k=0}^{+\infty} \frac{\left(-\frac{x^2}{2}\right)^k}{k!}$$

   We recognize the exponential series, with the input variable $-\frac{x^2}{2}$.

   Thus, for all $x \in]-R, R[$, $f(x) = e^{-x^2/2}$.

8. **(Bonus)** Check that the final expression that you got at previous question is a solution of (C) on the whole set $\mathbb{R}$.

   Let f be the function defined on $\mathbb{R}$ by $f(x) = e^{-x^2/2}$. Then for all $x \in \mathbb{R}$:

   - $f'(x) = \left(-x^2/2\right)' e^{-x^2/2} = -x\, e^{-x^2/2}$
   - $f''(x) = -e^{-x^2/2} - x\left(e^{-x^2/2}\right)' = -e^{-x^2/2} + x^2\, e^{-x^2/2}$

   Thus, $f(0) = 1$, $\quad f'(0) = 0 \quad$ and for all $x \in \mathbb{R}$,

   $$f''(x) + xf'(x) + f(x) = \left(-e^{-x^2/2} + x^2\, e^{-x^2/2}\right) - \left(x^2\, e^{-x^2/2}\right) + \left(e^{-x^2/2}\right) = 0$$

   f is hence a solution of (C).