EPITA

Mathematics

Midterm exam S3

November 2021

Duration: 3 hours

Name:

First name:

Class:

MARK:

Instructions:

- Documents and pocket calculators are not allowed.
- Write your answers on the stapled sheets provided for answering. No other sheet will be corrected.
- Please, do not use lead pencils for answering.

Exercise 1 (3 points)

1. Find the nature of the series whose general term is: $u_n = \dfrac{\sin(2n)}{n^2}$.

2. Find the nature of the series whose general term is: $u_n = \dfrac{n^2}{e^{n^2}}$.

3. What is the nature of the series $\sum \dfrac{(-1)^n}{\ln(n)}$?

Exercise 2 (3 points)

Consider the sequence (u_n) defined for all $n \in \mathbb{N}^*$ by: $u_n = \sqrt{n + (-1)^n} - \sqrt{n}$.

The purpose of the exercise is to study the nature of $\sum u_n$.

1. Find $(a, b) \in \mathbb{R}^2$ such that $u_n = \dfrac{a\,(-1)^n}{n^{\frac{1}{2}}} + \dfrac{b}{n^{\frac{3}{2}}} + o\left(\dfrac{1}{n^{\frac{3}{2}}}\right)$.

2. Using the previous question, determine the nature of $\sum u_n$.

Exercice 3 (4 points)

The purpose of this exercise is to study the nature of the sequence (u_n) defined by: $u_n = \dfrac{2 \times 4 \times 6 \times \cdots \times (2n)}{1 \times 3 \times 5 \times \cdots \times (2n-1)}$.

In that purpose, consider the auxiliary sequence (v_n) defined by: $v_n = \ln(u_n)$.

1. Let $n \in \mathbb{N}^*$. Compute $\dfrac{u_{n+1}}{u_n}$ and deduce $v_{n+1} - v_n$.

2. Find $a \in \mathbb{R}^+$ such that $(v_{n+1} - v_n) \sim \dfrac{a}{n}$.

3. What is the nature of (v_n)? Deduce what can be said about $\lim\limits_{n \to +\infty} v_n$.

4. Deduce $\lim\limits_{n \to +\infty} u_n$?

Exercise 4 (6 points)

In this exercise, the questions are mutually dependent.

If you have not answered to some of them, feel free to **accept their results without proof and to use them, if it is helpful, in other questions**.

1. Let $q \in \mathbb{R}^*$ and consider the power series $\sum q^n x^n$.

   a. What is the value of its radius of convergence R?

   b. Let f be the function defined on $]-R, R[$ by: $f(x) = \sum_{n=0}^{+\infty} q^n x^n$.

   Find an expression of $f(x)$ as a rational fraction.

   c. Deduce an expression of the function $g : x \longmapsto \dfrac{1}{(1-qx)^2}$ as a power series: $g(x) = \sum_{n=0}^{+\infty} a_n x^n$.

   Determine the coefficients a_n and the radius of convergence of the latter series.

2. Let $p \in]0, 1[$. Consider a random experiment which can lead to a success (with the probability p) or to a failure (with the probability $1 - p$). Assume that this experiment can be done as many times as you want, the outcomes being mutually independent.

   Finally, consider the random variable $X =$ « number of attempts required to get one success ».
   For example, if the first attempt is a success, then $X = 1$.

   a. Give the distribution of X.

b. Find the generating function of X. To start with, express $G_X(t)$ as a power series. Then express $G_X(t)$ as a rational fraction.

c. Deduce the expectation and the variance of X.

3. Consider the random variable $Y =$ « number of attempts required to get two successes ».

a. Express Y as a sum of two random variables studied previously.

b. Deduce its generating function $G_Y(t)$.

c. Application: using the previous questions, find $P(Y=5)$.

Exercise 5 (4 points)

Let X be an integer random variable whose generating function has the form $G_X(t) = a\ \ln\left(1 - \frac{t}{3}\right)$.

1. What is the value of a?

2. Using the geometric series, express $G_X(t)$ as a power series. Give the radius of convergence and deduce the distribution of X.

3. Find the expectation and the variance of X.

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4. Find a function f such that, if we define the random variable $Y = f(X)$, then: $G_Y(t) = t\,G_X(t)$.