

EPITA

Mathematics

Midterm exam S3

November 2021

Duration: 3 hours

Name:

First name:

Class:

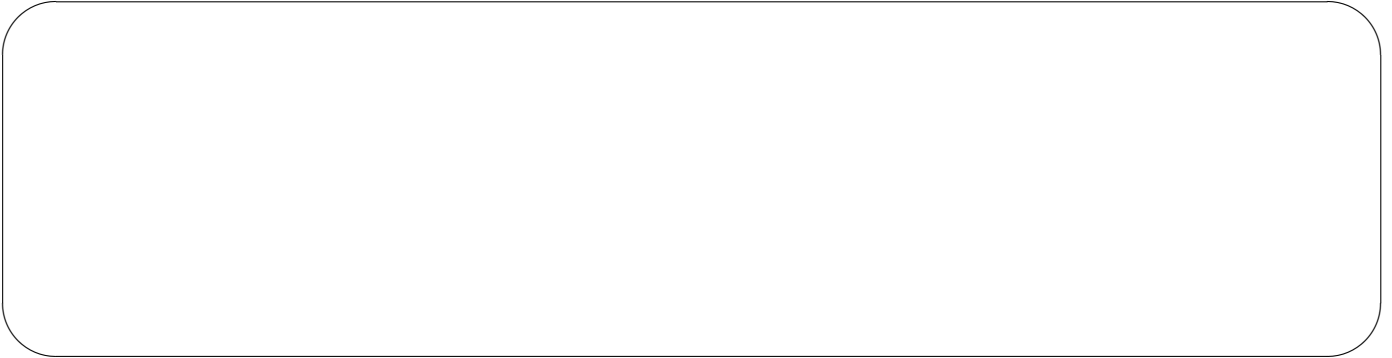
MARK:

Instructions:

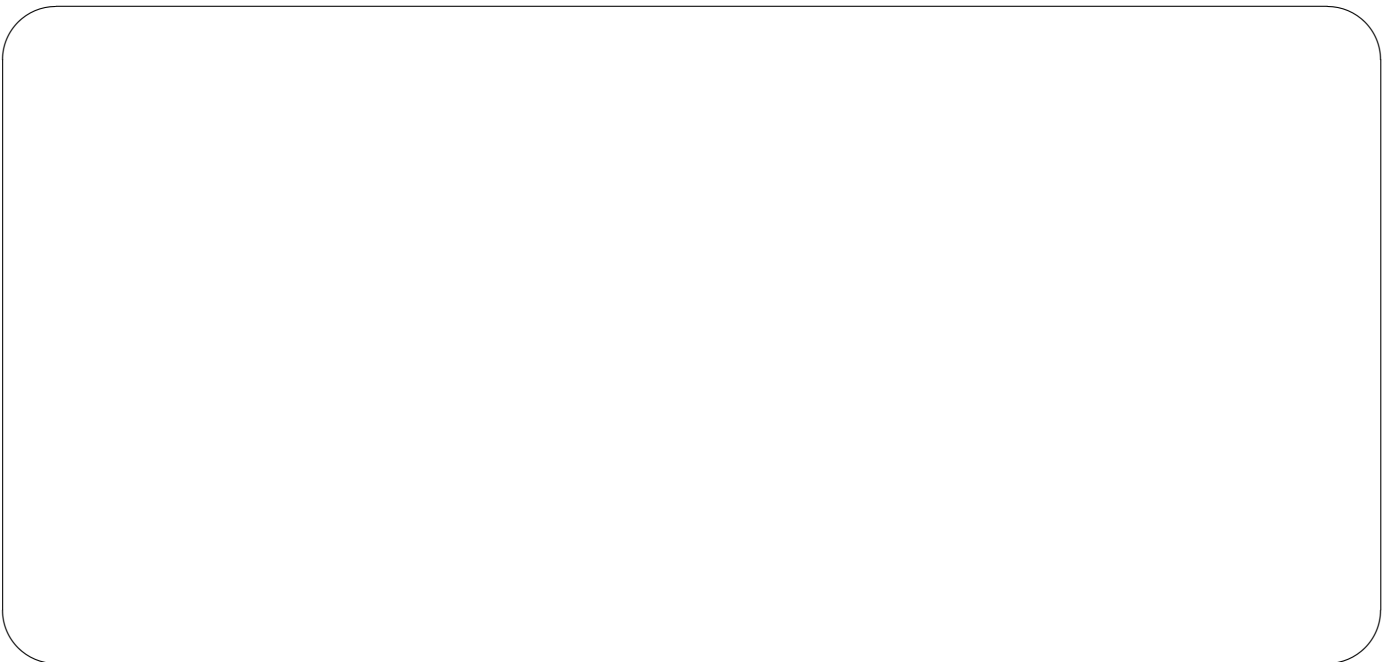
- Documents and pocket calculators are not allowed.
 - Write your answers on the stapled sheets provided for answering. No other sheet will be corrected.
 - Please, do not use lead pencils for answering.
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Exercise 1 (3 points)

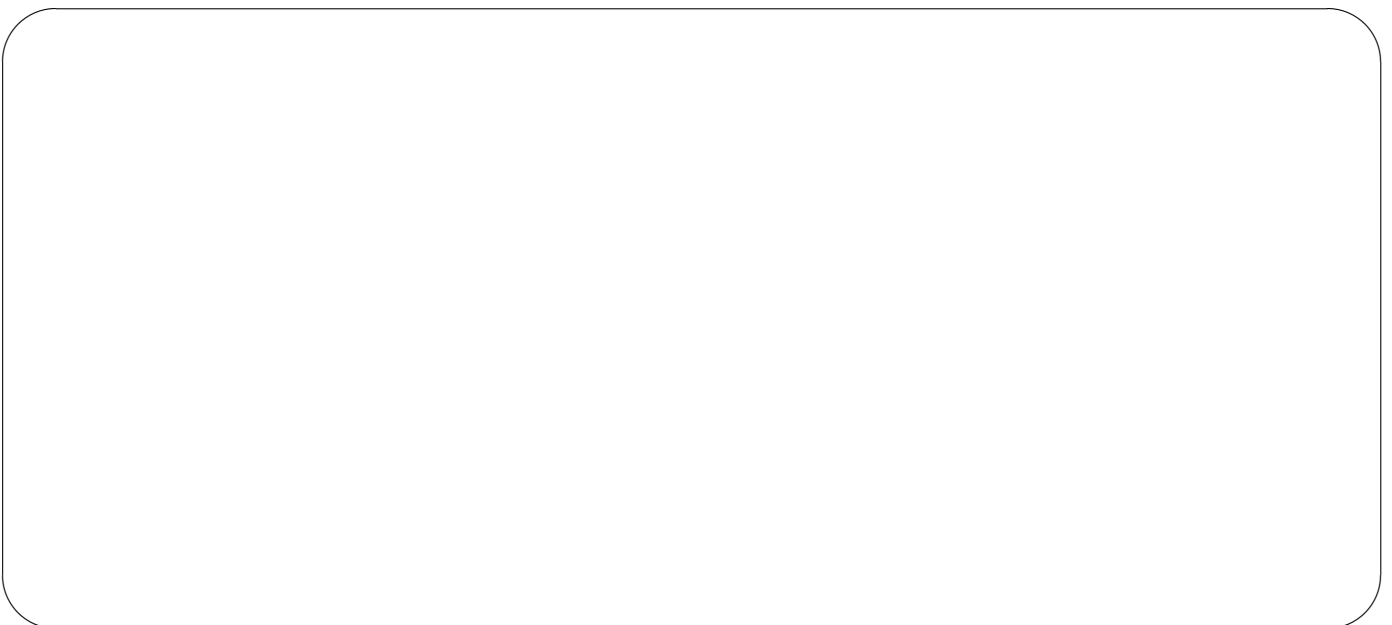
1. Find the nature of the series whose general term is: $u_n = \frac{\sin(2n)}{n^2}$.



2. Find the nature of the series whose general term is: $u_n = \frac{n^2}{e^{n^2}}$.



3. What is the nature of the series $\sum \frac{(-1)^n}{\ln(n)}$?



Exercise 2 (3 points)

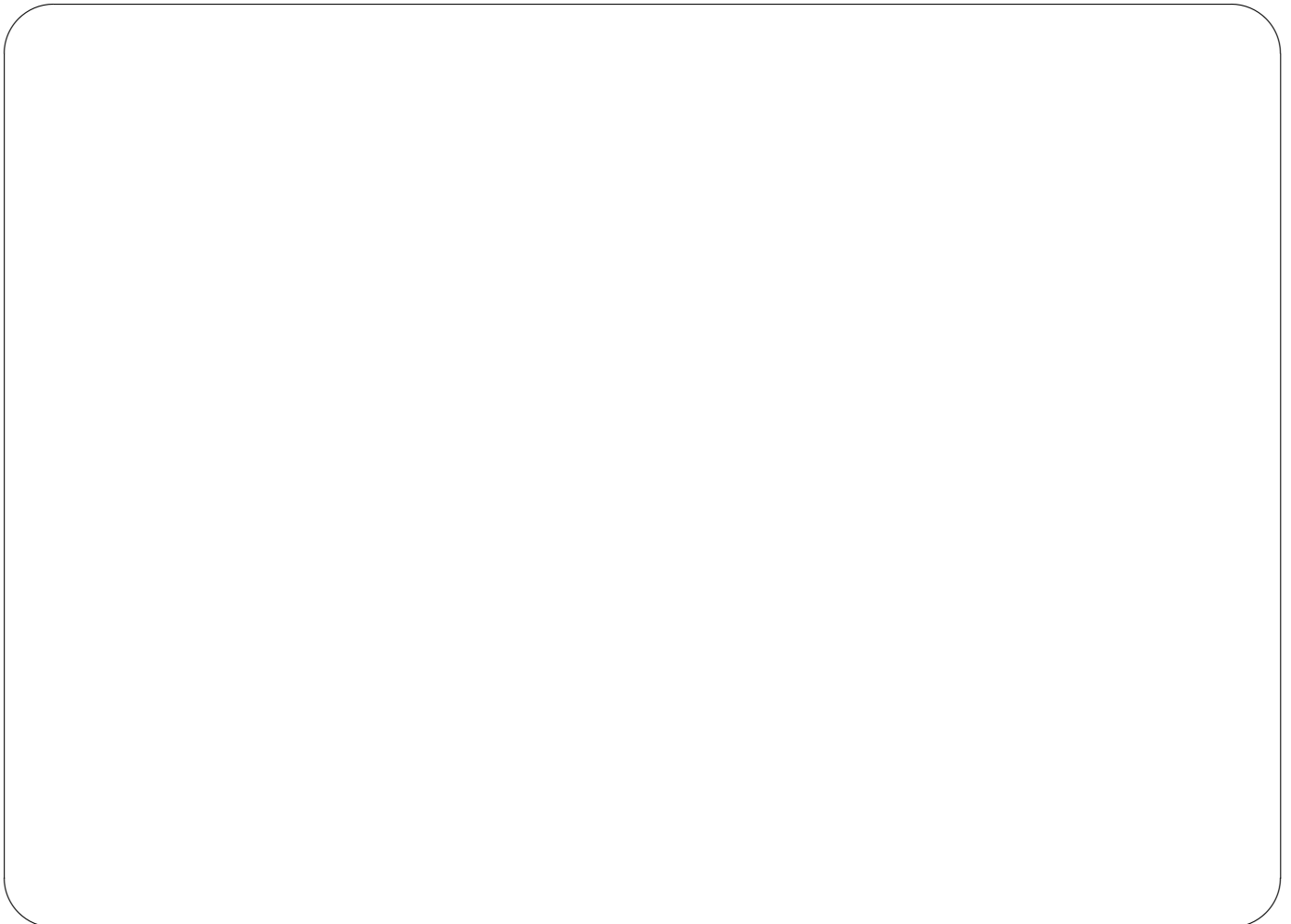
Consider the sequence (u_n) defined for all $n \in \mathbb{N}^*$ by: $u_n = \sqrt{n + (-1)^n} - \sqrt{n}$.

The purpose of the exercise is to study the nature of $\sum u_n$.

1. Find $(a, b) \in \mathbb{R}^2$ such that $u_n = \frac{a(-1)^n}{n^{\frac{1}{2}}} + \frac{b}{n^{\frac{3}{2}}} + o\left(\frac{1}{n^{\frac{3}{2}}}\right)$.



2. Using the previous question, determine the nature of $\sum u_n$.



Exercise 3 (4 points)

The purpose of this exercise is to study the nature of the sequence (u_n) defined by: $u_n = \frac{2 \times 4 \times 6 \times \dots \times (2n)}{1 \times 3 \times 5 \times \dots \times (2n-1)}$.

In that purpose, consider the auxiliary sequence (v_n) defined by: $v_n = \ln(u_n)$.

1. Let $n \in \mathbb{N}^*$. Compute $\frac{u_{n+1}}{u_n}$ and deduce $v_{n+1} - v_n$.

2. Find $a \in \mathbb{R}^+$ such that $(v_{n+1} - v_n) \sim \frac{a}{n}$.

3. What is the nature of (v_n) ? Deduce what can be said about $\lim_{n \rightarrow +\infty} v_n$.

4. Deduce $\lim_{n \rightarrow +\infty} u_n$?

Exercise 4 (6 points)

In this exercise, the questions are mutually dependent.

If you have not answered to some of them, feel free to **accept their results without proof and to use them, if it is helpful, in other questions.**

1. Let $q \in \mathbb{R}^*$ and consider the power series $\sum q^n x^n$.
- a. What is the value of its radius of convergence R ?

- b. Let f be the function defined on $] -R, R[$ by: $f(x) = \sum_{n=0}^{+\infty} q^n x^n$.

Find an expression of $f(x)$ as a rational fraction.

- c. Deduce an expression of the function $g : x \mapsto \frac{1}{(1 - qx)^2}$ as a power series: $g(x) = \sum_{n=0}^{+\infty} a_n x^n$.

Determine the coefficients a_n and the radius of convergence of the latter series.

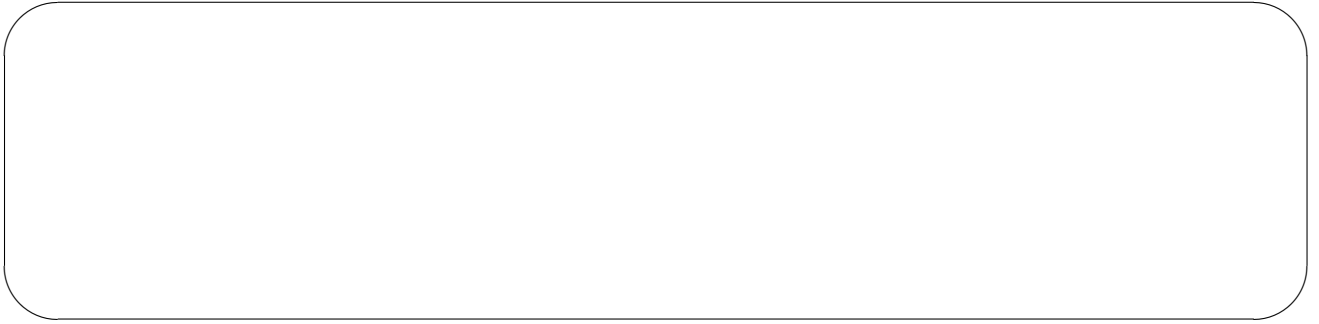
2. Let $p \in]0, 1[$. Consider a random experiment which can lead to a success (with the probability p) or to a failure (with the probability $1 - p$). Assume that this experiment can be done as many times as you want, the outcomes being mutually independent.

Finally, consider the random variable $X = \ll \text{number of attempts required to get one success} \gg$.

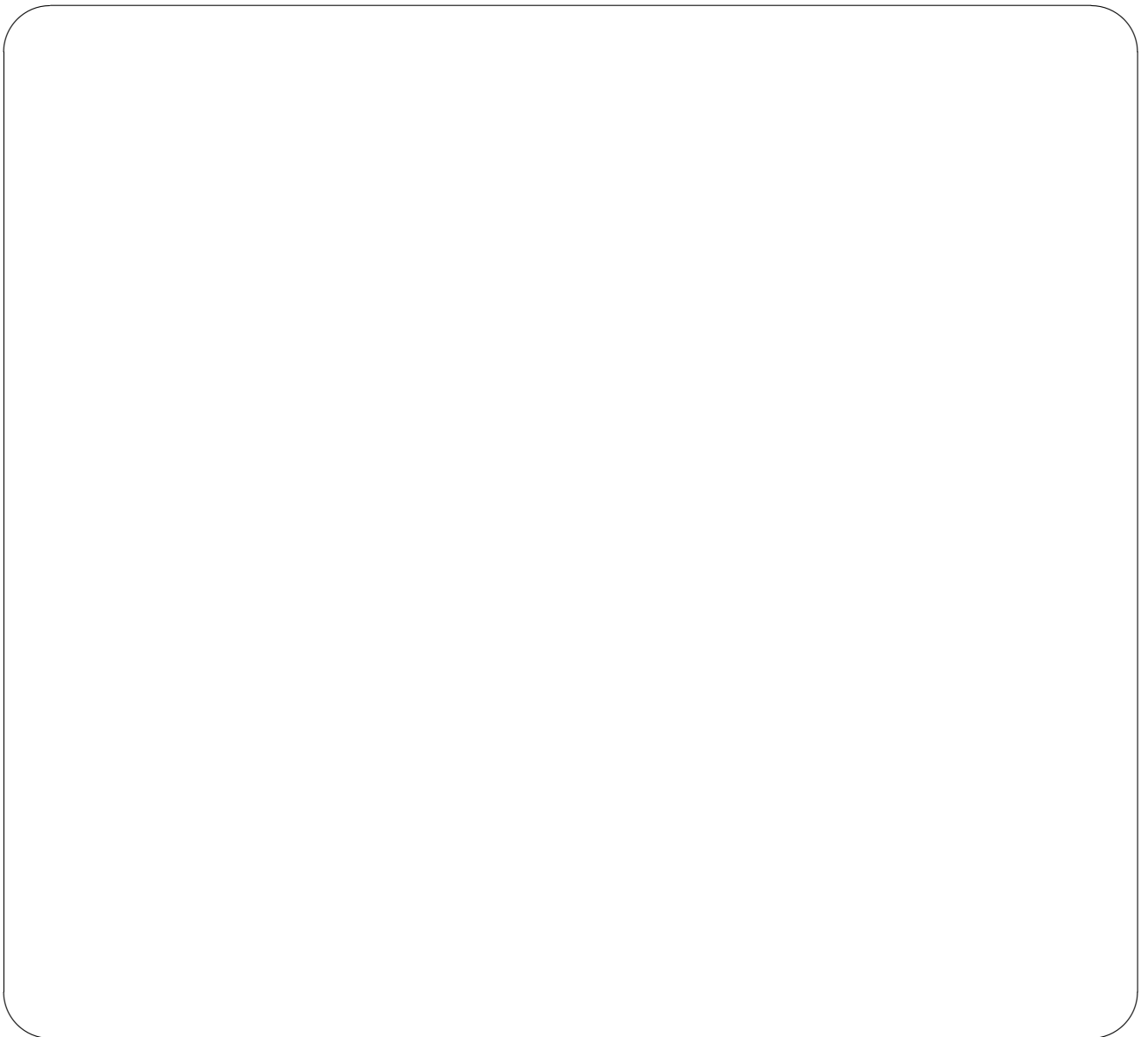
For example, if the first attempt is a success, then $X = 1$.

- a. Give the distribution of X .

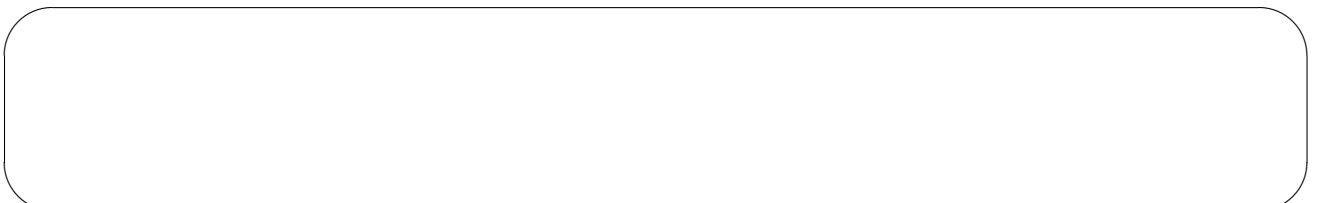
- b. Find the generating function of X . To start with, express $G_X(t)$ as a power series. Then express $G_X(t)$ as a rational fraction.



- c. Deduce the expectation and the variance of X .



3. Consider the random variable $Y =$ « number of attempts required to get two successes ».
a. Express Y as a sum of two random variables studied previously.



b. Deduce its generating function $G_Y(t)$.

c. Application: using the previous questions, find $P(Y=5)$.

Exercise 5 (4 points)

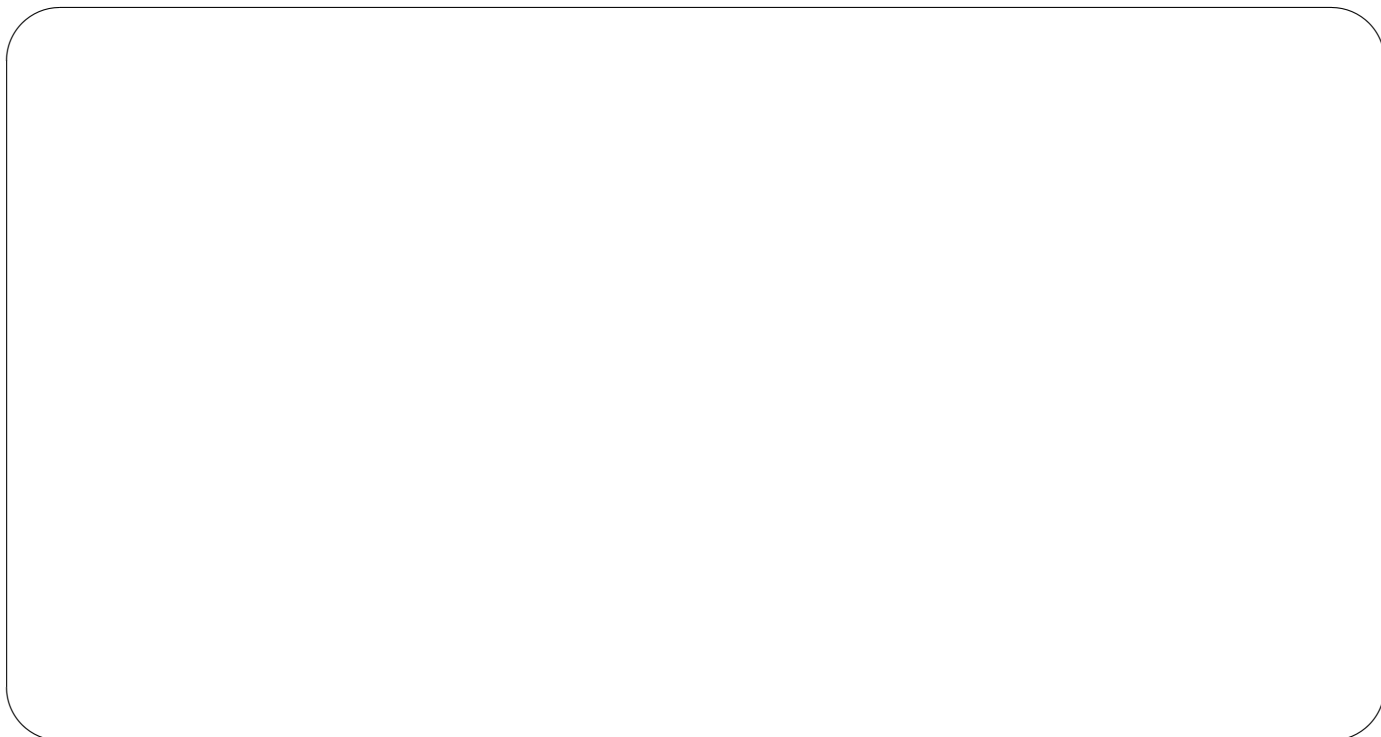
Let X be an integer random variable whose generating function has the form $G_X(t) = a \ln(1 - \frac{t}{3})$.

1. What is the value of a ?

2. Using the geometric series, express $G_X(t)$ as a power series. Give the radius of convergence and deduce the distribution of X .

3. Find the expectation and the variance of X .

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4. Find a function f such that, if we define the random variable $Y = f(X)$, then: $G_Y(t) = t G_X(t)$.

