EPITA

Mathematics

Midterm exam (S3)

October 2018

Name :

First name :

Class :

MARK :

Midterm exam S3

Duration : three hours
Documents and pocket calculators are not allowed

Exercise 1 (3 points)

1. Determine $\lim_{n\to+\infty} u_n$ where $u_n = n^2\left(e^{1/n^2} - \cos\left(\frac{1}{n}\right)\right)$.

2. Let $a \in \mathbb{R}^*$. Determine $\lim_{n\to+\infty} \left(1+\frac{1}{an}\right)^{2n}$.

Exercise 2 (5,5 points)

1. Determine $\lim\limits_{n \to +\infty} ne^{1/n} - n$ and then deduce the nature of the series $\sum \left(ne^{1/n} - n\right)$.

2. Let $a \in \mathbb{R}_+^*$. Using d'Alembert's rule (ratio test), determine the nature of the series $\sum \dfrac{(n!)^a}{(2n)!}$ depending on a.

3. Let $a \in]0,1[$. Using Cauchy's rule (root test), determine the nature of the series $\sum \frac{2^{\sqrt{n}}}{a^{n!}}$.

4. Let $a \in \mathbb{R}_+^*$. Determine the nature of $\sum \frac{(-1)^n}{n^a}$. Justify your answer.

Exercise 3 (6 points)

1. Let $N \in \mathbb{N}$, and let (u_n) and (v_n) be two strictly positive sequences such that, for any $n \geqslant N$, $\frac{u_{n+1}}{u_n} \leqslant \frac{v_{n+1}}{v_n}$.

   Prove that $\sum v_n$ convergent $\Longrightarrow \sum u_n$ convergent.

2. Let (u_n) be a strictly positive sequence such that $\dfrac{u_{n+1}}{u_n} = 1 - \dfrac{\alpha}{n} + o\left(\dfrac{1}{n}\right)$ where $\alpha \in \mathbb{R}$.

   a. Let $(v_n) = \left(\dfrac{1}{n^\beta}\right)$ where $\beta \in \mathbb{R}$. Show that $\dfrac{v_{n+1}}{v_n} = 1 - \dfrac{\beta}{n} + o\left(\dfrac{1}{n}\right)$.

   b. Suppose that $\alpha > 1$. Prove that $\sum u_n$ is convergent.

   N.B. : you may consider $\beta \in \mathbb{R}$ such that $1 < \beta < \alpha$ and use the sequence (v_n) defined in the previous question.

   c. Suppose that $\alpha < 1$. Prove that $\sum u_n$ is divergent.

   N.B. : you may consider $\beta \in \mathbb{R}$ such that $\alpha < \beta < 1$ and use the sequence (v_n) defined in the question a.

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3. What is the nature of $\sum u_n$ where $u_n = \dfrac{2 \times 4 \times \cdots \times 2n}{3 \times 5 \times \cdots \times (2n+1)}$.

4. Discuss, depending on the value of $a \in \mathbb{R}_+$, the nature of $\sum u_n$ where $u_n = \dfrac{n \times n!}{(a+1) \times \cdots \times (a+n)}$.

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Exercise 4 (3 points)

Let $\alpha \in \mathbb{R}_+^*$ and let $(u_n)_{n\geqslant 2}$ be the sequence defined for any $n \geqslant 2$ by $u_n = \dfrac{(-1)^n}{\sqrt{n^\alpha + (-1)^n}}$.

1. Verify that $u_n = \dfrac{(-1)^n}{n^{\alpha/2}} \cdot \dfrac{1}{\left(1 + \dfrac{(-1)^n}{n^\alpha}\right)^{1/2}}$.

2. Deduce $(a, b) \in \mathbb{R}^2$ such that $u_n = \dfrac{(-1)^n a}{n^{\alpha/2}} + \dfrac{b}{n^{3\alpha/2}} + o\left(\dfrac{1}{n^{3\alpha/2}}\right)$.

3. Deduce the nature of $\sum u_n$ depending on α.

Exercise 5 (3 points)

Determine the nature of the series $\sum u_n$ where, for any $n \in \mathbb{N}^*$, $u_n = \sqrt[3]{n^3 + 2n} - \sqrt{n^2 + 3}$.