EPITA

Mathematics

Midterm exam (S3)

November 2017

Name :

First name :

Class :

MARK :

Midterm exam (S3)

Duration : three hours
Documents and calculators not allowed

Exercise 1 (3 points)

1. Determine the Taylor expansion around 0 at order 2 of $e^x \ln(e + ex)$.

2. Determine $\lim_{x \to 0} \big(1 + \sin(x)\big)^{1/x}$.

Exercise 2 (5 points)

1. Determine $\lim_{n \to +\infty} \frac{\ln(n+1)}{\ln(n)}$. Then, using d'Alembert's rule, determine the nature of $\sum \frac{\ln(n)}{(n-1)!}$.

2. Determine the nature of the series $\sum u_n$ where $u_n = e - \left(1 + \frac{1}{n}\right)^n$.

3. Determine the nature of $\sum \left(\frac{(-1)^n}{\sqrt{n}} + \frac{1}{n} \right)$.

4. Determine the nature of $\sum \frac{\sin(n!)}{n^2}$.

Exercise 3 (5 points)

The purpose of this exercise is to determine the nature of the series with the general term : $u_n = (-1)^n n^\alpha \left(\ln\left(\frac{n+1}{n-1}\right)\right)^\beta$ where $(\alpha, \beta) \in \mathbb{R}^2$ and $n \in \mathbb{N}\backslash\{0,1\}$.

1. Show that $\ln\left(\frac{n+1}{n-1}\right) = \frac{2}{n}\left(1 + \frac{1}{3n^2} + o\left(\frac{1}{n^2}\right)\right)$.

2. Deduce that $u_n = (-1)^n \dfrac{2^\beta}{n^{\beta-\alpha}}\left(1 + \dfrac{\beta}{3n^2} + o\left(\dfrac{1}{n^2}\right)\right)$.

3. Show that in case $\beta \leqslant \alpha$, the series $\sum u_n$ diverges.

4. We focus on the case $\beta > \alpha$ and we put

$$u_n = (-1)^n \frac{2^\beta}{n^{\beta-\alpha}} + v_n \quad \text{with} \quad v_n = (-1)^n \frac{\beta 2^\beta}{3n^{2+\beta-\alpha}} + o\left(\frac{1}{n^{2+\beta-\alpha}}\right)$$

   a. Study the nature of the series $\sum v_n$.

   b. Study the nature of the series with the general term $w_n = (-1)^n \frac{2^\beta}{n^{\beta-\alpha}}$.

   c. Deduce the nature of $\sum u_n$.

Exercise 4 (3 points)

Let us consider the series $\sum u_n$ where $u_n = \left(\dfrac{n^2-3n+1}{n^2+n+1}\right)^{n^2}$.

1. Check that $\ln\left(\dfrac{n^2-3n+1}{n^2+n+1}\right) = \ln\left(1-\dfrac{3}{n}+\dfrac{1}{n^2}\right) - \ln\left(1+\dfrac{1}{n}+\dfrac{1}{n^2}\right)$.

2. Determine $a \in \mathbb{R}$ such that $\ln\left(\dfrac{n^2-3n+1}{n^2+n+1}\right) = \dfrac{a}{n} + o\left(\dfrac{1}{n}\right)$.

3. Deduce the nature of $\sum u_n$ using Cauchy's rule.

Exercise 5 (5 points)

Let $a \in \mathbb{R}$ and let $\sum u_n$ be the series with the general term $u_n = \left(\cos\left(\dfrac{1}{\sqrt{n}}\right)\right)^n - a$.

1. Using a Taylor expansion, determine $\lim\limits_{n\to+\infty}\left(\cos\left(\dfrac{1}{\sqrt{n}}\right)\right)^n$. Then deduce $\lim\limits_{n\to+\infty}\left(\cos\left(\dfrac{1}{\sqrt{n}}\right)\right)^n - a$.

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2. We suppose that $a \neq \dfrac{1}{\sqrt{e}}$. What can be said about the nature of $\sum u_n$?

3. We suppose now that $a = \dfrac{1}{\sqrt{e}}$.

   a. Using a Taylor expansion, show that $e^{n\ln(\cos(1/\sqrt{n}))} = e^{-\frac{1}{2}}\, e^{-\frac{1}{12n}+o\left(\frac{1}{n}\right)}$.

   b. Deduce an equivalent of u_n and the nature of $\sum u_n$.