# **EPITA**

## Mathematics

Midterm exam (S3)

November 2017

Name:
First name:
Class:

MARK:



# Midterm exam (S3)

Duration: three hours

Documents and calculators not allowed

Exercise	1	(3)	points)
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1.	1. Determine the Taylor expansion around 0 at order 2 of $e^x \ln(e + ex)$ .						
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2. Determine  $\lim_{x\to 0} (1+\sin(x))^{1/x}$ .

## Exercise 2 (5 points)

1. Determine  $\lim_{n\to+\infty}\frac{\ln(n+1)}{\ln(n)}$ . Then, using d'Alembert's rule, determine the nature of  $\sum \frac{\ln(n)}{(n-1)!}$ .

2. Determine the nature of the series  $\sum u_n$  where  $u_n = e - \left(1 + \frac{1}{n}\right)^n$ .

3. Determine the nature of  $\sum \left(\frac{(-1)^n}{\sqrt{n}} + \frac{1}{n}\right)$ .

4. Determine the nature of  $\sum \frac{\sin(n!)}{n^2}$ .

### Exercise 3 (5 points)

The purpose of this exercise is to determine the nature of the series with the general term :  $u_n = (-1)^n n^{\alpha} \left( \ln \left( \frac{n+1}{n-1} \right) \right)^{\beta}$  where  $(\alpha, \beta) \in \mathbb{R}^2$  and  $n \in \mathbb{N} \setminus \{0, 1\}$ .

1. Show that  $\ln\left(\frac{n+1}{n-1}\right) = \frac{2}{n}\left(1 + \frac{1}{3n^2} + o\left(\frac{1}{n^2}\right)\right)$ .

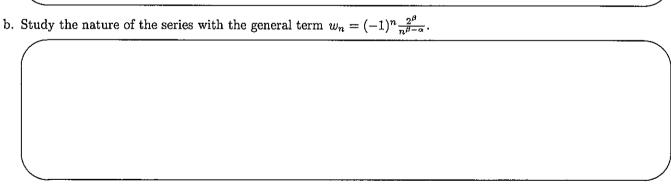
2. Deduce that  $u_n = (-1)^n \frac{2^{\beta}}{n^{\beta-\alpha}} \left(1 + \frac{\beta}{3n^2} + o\left(\frac{1}{n^2}\right)\right)$ .

3. Show that in case  $\beta \leqslant \alpha$ , the series  $\sum u_n$  diverges.

4. We focus on the case  $\beta > \alpha$  and we put

$$u_n = (-1)^n \frac{2^{\beta}}{n^{\beta - \alpha}} + v_n$$
 with  $v_n = (-1)^n \frac{\beta 2^{\beta}}{3n^{2+\beta - \alpha}} + o\left(\frac{1}{n^{2+\beta - \alpha}}\right)$ 

a. Study the nature of the series  $\sum v_n$ .



c. Deduce the nature of  $\sum u_n$ .

#### Exercise 4 (3 points)

Let us consider the series  $\sum u_n$  where  $u_n = \left(\frac{n^2 - 3n + 1}{n^2 + n + 1}\right)^{n^2}$ .

1. Check that  $\ln\left(\frac{n^2 - 3n + 1}{n^2 + n + 1}\right) = \ln\left(1 - \frac{3}{n} + \frac{1}{n^2}\right) - \ln\left(1 + \frac{1}{n} + \frac{1}{n^2}\right)$ .

2. Determine  $a \in \mathbb{R}$  such that  $\ln \left( \frac{n^2 - 3n + 1}{n^2 + n + 1} \right) = \frac{a}{n} + o\left( \frac{1}{n} \right)$ .

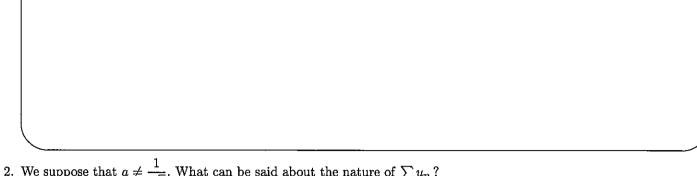
3. Deduce the nature of  $\sum u_n$  using Cauchy's rule.

#### Exercise 5 (5 points)

Let  $a \in \mathbb{R}$  and let  $\sum u_n$  be the series with the general term  $u_n = \left(\cos\left(\frac{1}{\sqrt{n}}\right)\right)^n - a$ .

1. Using a Taylor expansion, determine  $\lim_{n\to+\infty} \left(\cos\left(\frac{1}{\sqrt{n}}\right)\right)^n$ . Then deduce  $\lim_{n\to+\infty} \left(\cos\left(\frac{1}{\sqrt{n}}\right)\right)^n - a$ .

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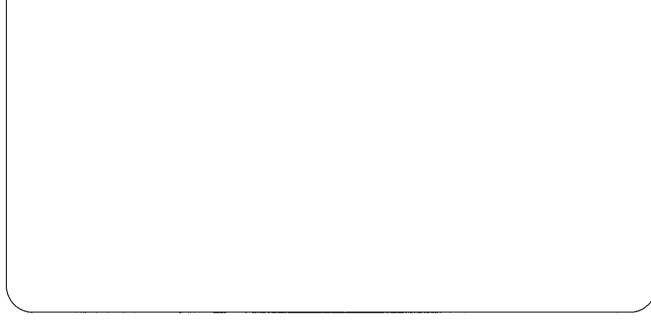


2. We suppose that  $a \neq \frac{1}{n}$ . What can be said about the nature of  $\sum u_n$ ?

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3. We suppose now that  $a = \frac{1}{\sqrt{e}}$ .

a. Using a Taylor expansion, show that  $e^{n \ln(\cos(1/\sqrt{n}))} = e^{-\frac{1}{2}} e^{-\frac{1}{12n} + o(\frac{1}{n})}$ .



b. Deduce an equivalent of  $u_n$  and the nature of  $\sum u_n$ .