Name : First Name : Class :

Midterm exam n°1

Exercise 1 (3 points)

1. Determine the Taylor expansion around 0 at order 3 of $\ln(2+\sin(x))$.

2. Determine $\lim\limits_{x\to+\infty}\left(\cos\left(\frac{1}{x}\right)\right)^{x^2}$.

Exercise 2 (5 points)

1. Using d'Alembert's test, determine the nature of $\sum \frac{(n!)^3}{(3n)!}$.

2. Using a Taylor expansion, determine the nature of $\sum \ln\left(\cos\left(\frac{1}{n}\right)\right)$.

3. Determine the nature of $\sum n^2 e^{-n}$.

4. Determine the nature of $\sum \frac{(-1)^n}{n\sqrt{n}}$.

Exercise 3 (4 points)

Let us consider the sequence $(u_n)_{n\in\mathbb{N}^*}$ defined for every $n \in \mathbb{N}^*$ by

$$u_n = \ln\big((n-1)!\big) - \left(n - \frac{1}{2}\right)\ln(n) + n$$

1. Show that

$$u_{n+1} - u_n = 1 - \left(n + \frac{1}{2}\right)\ln\left(1 + \frac{1}{n}\right)$$

2. Show that $u_{n+1} - u_n \underset{+\infty}{\sim} -\dfrac{1}{12n^2}$.

3. Deduce that (u_n) is convergent.

Exercise 4 (4 points)

Let us consider the sequence (u_n) defined for every $n \in \mathbb{N}^*$ by $u_n = \sqrt[n]{n+1} - \sqrt[n]{n}$.

1. Determine $\lim\limits_{n \to +\infty} \sqrt[n]{n}$.

2. Show that for every $n \in \mathbb{N}^*$, $u_n = \sqrt[n]{n}\left(\left(1+\frac{1}{n}\right)^{1/n} - 1\right)$.

3. Using a Taylor expansion, determine an equivalent of $\left(1+\frac{1}{n}\right)^{1/n} - 1$, then of u_n.

4. Deduce the nature of $\sum u_n$.

Exercise 5 (4 points)

Let us consider the sequence (u_n) defined for every $n \geqslant 2$ by $u_n = \dfrac{(-1)^n}{\ln(n) - (-1)^n}$.

1. What is the following limit : $\lim\limits_{n \to +\infty} \dfrac{n}{\ln^2(n)}$?

2. Deduce the nature of $\sum \dfrac{1}{\ln^2(n)}$.

3. Check that for every $n \in \mathbb{N}^*$, $u_n = \dfrac{(-1)^n}{\ln(n)} \left(1 - \dfrac{(-1)^n}{\ln(n)}\right)^{-1}$.

4. Determine $a \in \mathbb{R}$ such that $u_n = \dfrac{(-1)^n}{\ln(n)} + \dfrac{a}{\ln^2(n)} + o\left(\dfrac{1}{\ln^2(n)}\right)$.

5. Deduce the nature of $\sum u_n$.