

Midterm exam 1

Duration : three hours
Documents and calculators not allowed

Exercise 1 (2 points)

1. Determine the Taylor expansion around 0 at the order 3 of $\cos(x)^{\sin(x)}$.
2. Determine $\lim_{x \rightarrow 0} \frac{\ln(1 + \sin(x)) - \sin(\ln(1 + x))}{x^2 \sin(x^2)}$.

Exercise 2 (4,5 points)

1. Using the d'Alembert rule, determine the nature of $\sum \frac{2n}{n + 2^n}$.
2. Using the d'Alembert rule, determine the nature of $\sum \frac{1 + n^2}{n!}$.
3. Determine the nature of $\sum \frac{\sin(\sqrt{n} + 1)}{n^2}$.
4. Let $\alpha \in \mathbb{R}$. Determine, with precise and detailed arguments, the nature of $\sum \frac{(-1)^n}{n^\alpha}$.

Exercise 3 (8 points)

For all $n \in \mathbb{N}^*$, we set

$$a_n = \sum_{k=1}^n \frac{1}{\sqrt{k}}$$

1. a. What is the nature of the series $\sum \frac{1}{\sqrt{n}}$?
b. What is the limit of a_n when n tends to $+\infty$?
2. We consider the sequence $(u_n)_{n \geq 2}$ defined for all $n \geq 2$ by

$$u_n = \frac{(-1)^n}{(-1)^n + a_n}$$

- a. Show that for all $n \geq 2$,

$$u_n = \frac{(-1)^n}{a_n} - \frac{1}{a_n^2} + o\left(\frac{1}{a_n^2}\right)$$

- b. Show that the sequence $\left(\frac{1}{a_n}\right)_{n \in \mathbb{N}^*}$ is decreasing and converges to 0.
- c. Deduce the nature of the series $\sum \frac{(-1)^n}{a_n}$.
- d. Show by induction that, for all $n \in \mathbb{N}^*$,

$$2\sqrt{n+1} - 2 \leq a_n \leq 2\sqrt{n} - 1$$

N.B. : you can use the fact, without proving it, that for all $n \in \mathbb{N}^*$, $2\sqrt{n+2} - 2\sqrt{n+1} \leq \frac{1}{\sqrt{n+1}}$ and $2\sqrt{n+1} - 2\sqrt{n} \geq \frac{1}{\sqrt{n+1}}$.

e. Deduce the limit of $\frac{a_n}{2\sqrt{n}}$ when n tends to $+\infty$. Give an equivalent of a_n in $+\infty$.

f. Deduce the nature of the series $\sum \left(-\frac{1}{a_n^2} + o\left(\frac{1}{a_n^2}\right) \right)$.

3. Determine the nature of the series $\sum u_n$.

Exercise 4 (4,5 points)

We consider the sequence $(u_n)_{n \geq 2}$ such that

$$u_n = \ln \left(\frac{\sqrt{n} + (-1)^n}{\sqrt{n+1}} \right)$$

For $n \geq 2$, we set

$$v_n = \frac{\sqrt{n}}{\sqrt{n+1}}$$

1. Show that

$$u_n = \ln(v_n) + \ln \left(1 + \frac{(-1)^n}{\sqrt{n}} \right)$$

2. Determine $(\alpha, \beta) \in \mathbb{R}^2$ such that

$$v_n = 1 - \frac{\alpha}{n} + \frac{\beta}{n^2} + o\left(\frac{1}{n^2}\right)$$

3. Determine $\gamma \in \mathbb{R}$ such that

$$\ln(v_n) = -\frac{\alpha}{n} + \frac{\gamma}{n^2} + o\left(\frac{1}{n^2}\right)$$

4. Show that

$$u_n = \frac{(-1)^n}{\sqrt{n}} - \frac{1}{n} + \frac{(-1)^n}{3n\sqrt{n}} + o\left(\frac{1}{n\sqrt{n}}\right)$$

5. Deduce the nature of $\sum u_n$.

N.B. : your redaction at this last question is expected to be particularly precise and rigorous.

Exercise 5 (2 points)

1. Let (u_n) be a real sequence. Show that : $\sum(u_{n+1} - u_n)$ convergent $\iff (u_n)$ convergent.

2. Let $(a_n)_{n \in \mathbb{N}}$ be a real sequence with strictly positive terms and $u_0 \in \mathbb{R}_+^*$. We define $(u_n)_{n \in \mathbb{N}}$ by

$$\forall n \in \mathbb{N}, \quad u_{n+1} = u_n + \frac{a_n}{u_n}$$

Show that : (u_n) convergent $\iff \sum a_n$ convergent.

N.B. : you may use the fact (without proving it) that (u_n) is strictly positive and strictly increasing.