Correction of Midterm exam 1

Exercise 1 (2 points)

1. $\cos(x)^{\sin(x)} = e^{\sin(x)\ln\left(\cos(x)\right)}$.

   But

   $$\sin(x)\ln\left(\cos(x)\right) = \left(x - \frac{x^3}{6} + o(x^3)\right)\ln\left(1 - \frac{x^2}{2} + o(x^3)\right) = \left(x - \frac{x^3}{6} + o(x^3)\right)\left(-\frac{x^2}{2} + o(x^3)\right)$$

   so $\cos(x)^{\sin(x)} = e^{-x^3/2 + o(x^3)} = 1 - \dfrac{x^3}{2} + o(x^3)$.

2. $\ln\left(1+\sin(x)\right) = \ln\left(1 + x - \dfrac{x^3}{6} + o(x^4)\right) = \left(x - \dfrac{x^3}{6}\right) - \dfrac{1}{2}\left(x - \dfrac{x^3}{6}\right)^2 + \dfrac{1}{3}\left(x - \dfrac{x^3}{6}\right)^3 - \dfrac{1}{4}\left(x - \dfrac{x^3}{6}\right)^4 + o(x^4)$
   so

   $$\ln\left(1+\sin(x)\right) = x - \frac{x^3}{6} - \frac{1}{2}\left(x^2 - \frac{x^4}{3}\right) + \frac{1}{3}x^3 - \frac{1}{4}x^4 + o(x^4)$$

   Thus

   $$\ln\left(1+\sin(x)\right) = x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + o(x^4)$$

   $\sin\left(\ln(1+x)\right) = \sin\left(x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + o(x^4)\right) = \left(x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4}\right) - \dfrac{1}{6}\left(x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4}\right)^3 + o(x^4)$
   so

   $$\sin\left(\ln(1+x)\right) = x - \frac{x^2}{2} + \frac{x^3}{6} + o(x^4)$$

   Thus

   $$\ln\left(1+\sin(x)\right) - \sin\left(\ln(1+x)\right) = -\frac{x^4}{12} + o(x^4)$$

   and

   $$\ln\left(1+\sin(x)\right) - \sin\left(\ln(1+x)\right) \underset{0}{\sim} -\frac{x^4}{12}$$

   On the other hand, $x^2\sin(x^2) \underset{0}{\sim} x^4$. Thus

   $$\frac{\ln\left(1+\sin(x)\right) - \sin\left(\ln(1+x)\right)}{x^2\sin(x^2)} \underset{0}{\sim} \frac{-\frac{x^4}{12}}{x^4} = -\frac{1}{12}$$

   and

   $$\lim_{x\to 0}\left[\frac{\ln\left(1+\sin(x)\right) - \sin\left(\ln(1+x)\right)}{x^2\sin(x^2)}\right] = -\frac{1}{12}$$

Exercice 2 (4,5 points)

1. Let's call $u_n = \dfrac{2n}{n+2^n}$.

   $$\begin{aligned}\frac{u_{n+1}}{u_n} &= \frac{2n+2}{n+1+2^{n+1}} \times \frac{n+2^n}{2n} \\ &= \frac{n+1}{n} \times \frac{2^n}{2^{n+1}} \times \frac{\frac{n}{2^n}+1}{\frac{n+1}{2^{n+1}}+1} \xrightarrow[n\to+\infty]{} \frac{1}{2}\end{aligned}$$

   But $\dfrac{1}{2} < 1$ so, via the rule of d'Alembert, $\sum u_n$ is convergent.

2. Let's call $v_n = \dfrac{1+n^2}{n!}$.

   $$\frac{v_{n+1}}{v_n} = \frac{1+(n+1)^2}{(n+1)!} \times \frac{n!}{1+n^2} = \frac{1}{n+1} \times \frac{n^2+2n+2}{n^2+1} \xrightarrow[n\to+\infty]{} 0.$$

   But $0 < 1$ so $\sum v_n$ is convergent.

3. $\left|\dfrac{\sin\left(\sqrt{n}+1\right)}{n^2}\right| \leqslant \dfrac{1}{n^2}$ and $\sum \dfrac{1}{n^2}$ is convergent so $\sum \dfrac{\sin\left(\sqrt{n}+1\right)}{n^2}$ is absolutely convergent. Therefore it is convergent.

4. If $\alpha \leqslant 0$, the term of the series does not tend to 0 so $\sum \dfrac{(-1)^n}{n^\alpha}$ is divergent.

   If $\alpha > 1$, $\sum \dfrac{(-1)^n}{n^\alpha}$ is absolutely convergent so it is convergent.

   If $0 < \alpha \leqslant 1$, the series $\sum \dfrac{(-1)^n}{n^\alpha}$ is alternating and satisfies the conditions of the alternating series criterion : the sequence $\left(\dfrac{1}{n^\alpha}\right)$ is decreasing and tends to 0 so $\sum \dfrac{(-1)^n}{n^\alpha}$ is convergent.

Exercise 3 (8 points)

1. a. $\sum \dfrac{1}{\sqrt{n}}$ is a divergent Riemann series because $\dfrac{1}{2} < 1$.

   b. (a_n) is the sequence of partial sums associated to the series $\sum \dfrac{1}{\sqrt{n}}$. It is increasing because the series has positive terms. Via the previous question, $\sum \dfrac{1}{\sqrt{n}}$ is divergent so a_n diverges to $+\infty$ when n tends to $+\infty$.

2. a. We have

   $$\begin{aligned} u_n &= \frac{(-1)^n}{a_n}\left(1+\frac{(-1)^n}{a_n}\right)^{-1} \\ &= \frac{(-1)^n}{a_n}\left(1-\frac{(-1)^n}{a_n}+o\left(\frac{1}{a_n}\right)\right) \\ &= \frac{(-1)^n}{a_n} - \frac{1}{a_n^2} + o\left(\frac{1}{a_n^2}\right) \end{aligned}$$

   b. $\dfrac{1}{a_{n+1}} - \dfrac{1}{a_n} = \dfrac{a_n - a_{n+1}}{a_n a_{n+1}}$

   but

   $$a_n - a_{n+1} = -\frac{1}{\sqrt{n+1}} < 0$$

   so $\dfrac{1}{a_{n+1}} - \dfrac{1}{a_n} < 0$. Thus, $\left(\dfrac{1}{a_n}\right)_{n\in\mathbb{N}^*}$ is decreasing and, according to $1.b$, converges to 0.

   c. The series $\sum \dfrac{(-1)^n}{a_n}$ is convergent according the alternating series criterion.

   d. We have

   $$2\sqrt{2} - 2 \leqslant a_1 = 1 \leqslant 2\sqrt{1} - 1 = 1$$

   so the property is true for $n = 1$. Suppose now that it is true for a given value $n \geqslant 1$. Then, since

   $$a_{n+1} = a_n + \frac{1}{\sqrt{n+1}}$$

we have, via the induction hypothesis and the remark in N.B.,

$$2\sqrt{n+1}-2+2\sqrt{n+2}-2\sqrt{n+1}\leqslant a_{n+1}\leqslant 2\sqrt{n}-1+2\sqrt{n+1}-2\sqrt{n}$$

that is

$$2\sqrt{n+2}-2\leqslant a_{n+1}\leqslant 2\sqrt{n+1}-1$$

Finally, the property is also true for $n+1$.

e. Refering to the previous question, we have

$$\frac{\sqrt{n+1}-1}{\sqrt{n}}\leqslant\frac{a_n}{2\sqrt{n}}\leqslant\frac{2\sqrt{n}-1}{2\sqrt{n}}$$

which means that

$$\sqrt{1+\frac{1}{n}}-\frac{1}{\sqrt{n}}\leqslant\frac{a_n}{2\sqrt{n}}\leqslant 1-\frac{1}{2\sqrt{n}}$$

Using the squeeze theorem, we deduce that $\frac{a_n}{2\sqrt{n}}\longrightarrow 1$. Hence

$$a_n\underset{+\infty}{\sim}2\sqrt{n}$$

f. According to the previous question,

$$-\frac{1}{a_n^2}+o\left(\frac{1}{a_n^2}\right)\underset{+\infty}{\sim}-\frac{1}{a_n^2}\underset{+\infty}{\sim}-\frac{1}{4n}$$

but $\sum\frac{1}{n}$ is divergent so $\sum\left(-\frac{1}{a_n^2}+o\left(\frac{1}{a_n^2}\right)\right)$ is divergent.

3. $\sum u_n$ is the sum of a convergent series and a divergent one. It is therefore divergent.

Exercise 4 (4,5 points)

1.
$$\begin{aligned} u_n &= \ln\left(\frac{\sqrt{n}+(-1)^n}{\sqrt{n+1}}\right)=\ln\left(\frac{\sqrt{n}}{\sqrt{n+1}}+\frac{(-1)^n}{\sqrt{n+1}}\right)=\ln\left(\frac{\sqrt{n}}{\sqrt{n+1}}\left(1+\frac{(-1)^n}{\sqrt{n}}\right)\right)\\ &= \ln\left(\frac{\sqrt{n}}{\sqrt{n+1}}\right)+\ln\left(1+\frac{(-1)^n}{\sqrt{n}}\right)\\ &= \ln(v_n)+\ln\left(1+\frac{(-1)^n}{\sqrt{n}}\right)\end{aligned}$$

2.
$$\begin{aligned} v_n &= \sqrt{\frac{n}{n+1}}=\left(1+\frac{1}{n}\right)^{-\frac{1}{2}}\\ &= 1-\frac{1}{2n}+\frac{3}{8n^2}+o\left(\frac{1}{n^2}\right)\end{aligned}$$

So $\alpha=\frac{1}{2}$ and $\beta=\frac{3}{8}\cdot$

3. $$\ln(v_n) = \ln\left(1 - \frac{1}{2n} + \frac{3}{8n^2} + o\left(\frac{1}{n^2}\right)\right) = -\frac{1}{2n} + \frac{3}{8n^2} - \frac{1}{2}\left(\frac{1}{2n} - \frac{3}{8n^2}\right)^2 + o\left(\frac{1}{n^2}\right)$$
$$= -\frac{1}{2n} + \frac{3}{8n^2} - \frac{1}{8n^2} + o\left(\frac{1}{n^2}\right)$$
$$= -\frac{1}{2n} + \frac{1}{4n^2} + o\left(\frac{1}{n^2}\right)$$
So $\gamma = \frac{1}{4}$.

4. $$u_n = \ln(v_n) + \ln\left(1 + \frac{(-1)^n}{\sqrt{n}}\right) = -\frac{1}{2n} + \frac{1}{4n^2} + o\left(\frac{1}{n^2}\right) + \ln\left(1 + \frac{(-1)^n}{\sqrt{n}}\right)$$
But
$$\ln\left(1 + \frac{(-1)^n}{\sqrt{n}}\right) = \frac{(-1)^n}{\sqrt{n}} - \frac{(-1)^{2n}}{2n} + \frac{(-1)^{3n}}{3n\sqrt{n}} + o\left(\frac{1}{n\sqrt{n}}\right)$$
$$= \frac{(-1)^n}{\sqrt{n}} - \frac{1}{2n} + \frac{(-1)^n}{3n\sqrt{n}} + o\left(\frac{1}{n\sqrt{n}}\right)$$
Furthermore
$$\frac{1}{4n^2} + o\left(\frac{1}{n^2}\right) = o\left(\frac{1}{n\sqrt{n}}\right)$$
Finally
$$u_n = -\frac{1}{2n} + \frac{(-1)^n}{\sqrt{n}} - \frac{1}{2n} + \frac{(-1)^n}{3n\sqrt{n}} + o\left(\frac{1}{n\sqrt{n}}\right)$$
$$= \frac{(-1)^n}{\sqrt{n}} - \frac{1}{n} + \frac{(-1)^n}{3n\sqrt{n}} + o\left(\frac{1}{n\sqrt{n}}\right)$$

5. $u_n = \frac{(-1)^n}{\sqrt{n}} - \frac{1}{n} + \frac{(-1)^n}{3n\sqrt{n}} + o\left(\frac{1}{n\sqrt{n}}\right)$

But $\sum \frac{(-1)^n}{\sqrt{n}}$ is alternating and convergent, because $\left(\left|\frac{(-1)^n}{\sqrt{n}}\right|\right)$ is decreasing and tends to 0.

Let's call
$$w_n = \frac{(-1)^n}{3n\sqrt{n}} + o\left(\frac{1}{n\sqrt{n}}\right)$$
Then
$$|w_n| \underset{+\infty}{\sim} \frac{1}{3n\sqrt{n}} = \frac{1}{3n^{\frac{3}{2}}}$$
and $\sum \frac{1}{n^{\frac{3}{2}}}$ is convergent. So $\sum w_n$ is absolutely convergent and is therefore convergent.

On the other hand, $\sum \frac{1}{n}$ is divergent.

Finally $\sum u_n$, being the sum of a divergent series and a convergent one, is divergent.

Exercise 5 (2 points)

1. $\sum(u_{n+1}-u_n)$ convergent $\iff \left(\sum_{k=0}^{n}(u_{k+1}-u_k)\right)$ convergent $\iff (u_n-u_0)$ convergent so

$$\sum(u_{n+1}-u_n) \text{ convergent } \iff (u_n)=(u_n-u_0+u_0) \text{ convergent}$$

2. $\boxed{\Longleftarrow}$

Suppose that $\sum a_n$ is convergent. Then

$$\forall n\in\mathbb{N},\ 0<u_{n+1}-u_n=\frac{a_n}{u_n}<\frac{a_n}{u_0}$$

so $\sum(u_{n+1}-u_n)$ is convergent by comparaison and, according to the question 1, (u_n) is convergent.

$\boxed{\Longrightarrow}$

Suppose now that (u_n) converges to ℓ. Since (u_n) is strictly increasing and strictly positive, $\ell>0$. Furtheremore

$$0<u_{n+1}-u_n=\frac{a_n}{u_n}\underset{+\infty}{\sim}\frac{a_n}{\ell}$$

which implies that

$$a_n\underset{+\infty}{\sim}\ell(u_{n+1}-u_n)$$

But (u_n) is convergent so, according to the question 1, $\sum(u_{n+1}-u_n)$ is convergent.

Thus, $\sum a_n$ is convergent by comparaison.