Exercise 1

1. Using D'Alembert's test (ratio test), determine the nature of the series $\sum \frac{n!}{n^n}$.
2. Using Cauchy's test (root test), determine the nature of the series $\sum \left(\frac{(n+1)^2}{(an)^2+1} \right)^n$ depending on the parameter $a \in \mathbb{R}$.
3. Using Leibniz's test for alternating series, determine the nature of the series $\sum (-1)^n \frac{n+1}{n \ln(n)}$.

Exercise 2

Let $A = \begin{pmatrix} 1 & -4 & -2 \\ -1 & 1 & -1 \\ 2 & 4 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -2 & -2 \\ -2 & -1 & -4 \\ 2 & 4 & 7 \end{pmatrix}$.

Are A and B diagonalisable in $\mathscr{M}_3(\mathbb{R})$? If so, give a transfer matrix and the associated diagonal matrix ; find the eigenvectors using a methodic study of the eigenspaces.

Exercise 3

Study, depending on the parameter $a \in \mathbb{R}$, the diagonalisability of the matrix

$$A = \begin{pmatrix} 1 & 2-2a & 1-a \\ 1 & 4 & 1 \\ 0 & 2a-2 & a \end{pmatrix}$$

(It is not necessary to give a decomposition with transfer matrices and a diagonal matrix).

Exercise 4

Let A be the matrix $\begin{pmatrix} 4 & 4 & 2 \\ 4 & 3 & 3 \\ 4 & 5 & 1 \end{pmatrix}$.

We denote f the endomorphism of $\mathbb{R}^3$ standardly associated to A (that is to say, if we denote $\mathscr{B}$ the standard basis of $\mathbb{R}^3$ then $A = \text{Mat}_{\mathscr{B}}(f)$).
Let $\mathscr{E}$ be the set $\big((1,0,1),(2,2,2),(3,3,1)\big)$ of vectors from $\mathbb{R}^3$.

1. Determine $\text{Ker}(f)$ and $\text{Im}(f)$.
2. Is A invertible ? Justify without calculations.
3. Show that $\mathscr{E}$ is a basis of $\mathbb{R}^3$.
4. Determine $\text{Mat}_{\mathscr{E}}(f)$

Exercise 5

Let E be a vector space over $\mathbb{R}$, of dimension 4, and let $\mathscr{E} = (e_1, e_2, e_3, e_4)$ be a basis of E.
Let $p \in \mathscr{L}(E)$ such that

$$\mathrm{Mat}_{\mathscr{E}}(p) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

1. Show that p is a projection (i.e. $p \circ p = p$).
2. By looking at the images by p of the vectors of $\mathscr{E}$, find a basis of $\mathrm{Im}(p)$.
3. Using the rank theorem, find the dimension of $\mathrm{Ker}(p)$.
4. Deduce a basis of $\mathrm{Ker}(p)$ depending on the e_i vectors.

Exercise 6

The goal of this exercise is to determine a direct formula to compute the terms of the sequence defined by the relation of recurrence

$$u_{n+3} = -u_{n+2} + 4u_{n+1} + 4u_n$$

and whose first terms are $u_0 = 0, u_1 = 1, u_2 = 1$.

1. Denoting $X_n = \begin{pmatrix} u_{n+2} \\ u_{n+1} \\ u_n \end{pmatrix}$, determine a matrix M such that $X_{n+1} = MX_n$. Find an expression of X_n depending on M, n and X_0.
2. Calculate (over expanded form) the characteristic polynomial of M ; by remarking that it can be divided by $(X+1)$, factorise it. Show that M is diagonalisable, and find a diagonal matrix D and an invertible matrix P such that $M = PDP^{-1}$.
3. Deduce M^n depending on n, then u_n depending on n.

   Remark : you can check the compatibility of your results with the given data by comparing the first values of u_n, calculated both with the relation of recurrence and the obtained formula.