Exercise 1

1. Let us denote $u_n = \dfrac{n!}{n^n}$.

$$\frac{u_{n+1}}{u_n} = \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \frac{(n+1)!}{n!} \cdot \frac{n^n}{(n+1)^{n+1}} = \frac{(n+1)n^n}{(n+1)(n+1)^n} = \left(\frac{n}{n+1}\right)^n = \left(1+\frac{1}{n}\right)^{-n}$$
$$= \mathrm{e}^{-n\ln\left(1+\frac{1}{n}\right)} = \mathrm{e}^{-n\left(\frac{1}{n}+o\left(\frac{1}{n}\right)\right)} = \mathrm{e}^{-1+o(1)} \longrightarrow \frac{1}{\mathrm{e}}$$

$\dfrac{1}{\mathrm{e}} < 1$ hence, using D'Alembert's test, $\sum u_n$ converges.

2. Let us denote $v_n = \left(\dfrac{(n+1)^2}{(an)^2+1}\right)^n$.

$$\sqrt[n]{v_n} = \frac{(n+1)^2}{(an)^2+1} : \quad \left| \begin{array}{l} \text{If } a=0,\ \sqrt[n]{v_n} = (n+1)^2 \underset{n\to+\infty}{\longrightarrow} +\infty \\ \text{If } a\neq 0,\ \sqrt[n]{v_n} \underset{+\infty}{\sim} \dfrac{n^2}{(an)^2} = \dfrac{1}{a^2} \end{array} \right.$$

Then, using Cauchy's test :
— if $a=0$ then $\sum v_n$ diverges ;
— if $\dfrac{1}{a^2} > 1$ i.e. $a \in \,]-1;1[\, \setminus\{0\}$ then $\sum v_n$ diverges ;
— if $\dfrac{1}{a^2} < 1$ i.e. $a \in \,]-\infty;-1[\, \cup \,]1;+\infty[$ then $\sum v_n$ converges.

If $a\in\{-1;1\}$ we cannot conclude using Cauchy's test. However, when this happens :

$$v_n = \left(\frac{n^2+2n+1}{n^2+1}\right)^n = \left(1+\frac{2n}{n^2+1}\right)^n > 1 \text{ thus } v_n \nrightarrow 0$$

As v_n does not respect the necessary condition of convergence, $\sum v_n$ diverges.

3. Let us denote $w_n = \dfrac{n+1}{n\ln(n)}$.

$(w_n)_{n\geqslant 2}$ is a sequence of strictly positive terms, and $w_n \underset{+\infty}{\sim} \dfrac{1}{\ln(n)} \underset{+\infty}{\longrightarrow} 0$.

Let us show that (w_n) is decreasing, by studying the variations of the function $f : x \longmapsto \dfrac{x+1}{x\ln(x)}$.

Over $[2,+\infty[$:

$$f'(x) = \frac{x\ln(x) - (x+1)\big(\ln(x)+1\big)}{x^2\ln^2(x)} = \frac{-x-1-\ln(x)}{x^2\ln^2(x)} < 0.$$

f is thus strictly decreasing over $[2,+\infty[$, thus $(w_n) = \big(f(n)\big)$ is also strictly decreasing.
Then, using Leibniz's test for alternating series :
w_n is decreasing and tends towards 0, thus $\sum(-1)^n w_n$ converges.

Exercise 2

$$\begin{aligned}
P_A(X) &= \begin{vmatrix} 1-X & -4 & -2 \\ -1 & 1-X & -1 \\ 2 & 4 & 5-X \end{vmatrix} \\
&\underset{L_1 \leftarrow L_1 + L_3}{=} \begin{vmatrix} 3-X & 0 & 3-X \\ -1 & 1-X & -1 \\ 2 & 4 & 5-X \end{vmatrix} \\
&\underset{C_3 \leftarrow C_3 - C_1}{=} \begin{vmatrix} 3-X & 0 & 0 \\ -1 & 1-X & 0 \\ 2 & 4 & 3-X \end{vmatrix} \\
&= (1-X)(3-X)^2
\end{aligned}$$

Thus, P_A is split, and the eigenvalues of A are 1 (of multiplicity 1) and 3 (of multiplicity 2). As we necessarily have $\dim(E_1) = 1$, A will be diagonalisable if and only if $\dim(E_3) = 2$.

$$\begin{aligned}
\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in E_3 \Longleftrightarrow A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 3 \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad &\Longleftrightarrow \left\{ \begin{array}{rrrcl} x & -4y & -2z & = & 3x \\ -x & +y & -z & = & 3y \\ 2x & +4y & +5z & = & 3z \end{array} \right. \\
&\Longleftrightarrow \left\{ \begin{array}{rrrcl} -2x & -4y & -2z & = & 0 \\ -x & -2y & -z & = & 0 \\ 2x & +4y & +2z & = & 0 \end{array} \right. \\
&\Longleftrightarrow x + 2y + z = 0
\end{aligned}$$

From which we can deduce that $E_3 = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \right\}$; thus, as $\dim(E_3) = 2 = m(3)$ the matrix A is diagonalisable.

Let us look for an eigenvector associated with the eigenvalue 1 to build a transfer matrix :

$$\begin{aligned}
\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in E_1 \Longleftrightarrow A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad &\Longleftrightarrow \left\{ \begin{array}{rrrcl} x & -4y & -2z & = & x \\ -x & +y & -z & = & y \\ 2x & +4y & +5z & = & z \end{array} \right. \\
&\Longleftrightarrow \left\{ \begin{array}{rrrcl} & -4y & -2z & = & 0 \\ -x & & -z & = & 0 \\ 2x & +4y & +4z & = & 0 \end{array} \right. \\
&\Longleftrightarrow \left\{ \begin{array}{rcl} x & = & -z \\ z & = & -2y \end{array} \right.
\end{aligned}$$

Thus, $E_1 = \text{Span}\left\{ \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} \right\}$.

Therefore :

$A = PDP^{-1}$ with for instance $P = \begin{pmatrix} 2 & 1 & 2 \\ 1 & 0 & -1 \\ -2 & -1 & 0 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

$$\begin{aligned}
P_B(X) &= \begin{vmatrix} 1-X & -2 & -2 \\ -2 & -1-X & -4 \\ 2 & 4 & 7-X \end{vmatrix} \\
&\underset{L_2 \leftarrow L_2+L_3}{=} \begin{vmatrix} 1-X & -2 & -2 \\ 0 & 3-X & 3-X \\ 2 & 4 & 7-X \end{vmatrix} \\
&\underset{C_3 \leftarrow C_3-C_2}{=} \begin{vmatrix} 1-X & -2 & 0 \\ 0 & 3-X & 0 \\ 2 & 4 & 3-X \end{vmatrix} \\
&= (3-X) \begin{vmatrix} 1-X & -2 \\ 0 & 3-X \end{vmatrix} \\
&= (1-X)(3-X)^2
\end{aligned}$$

P_B is split, and the eigenvalues of B are thus also 1 (of multiplicity 1) and 3 (of multiplicity 2). As for A, we already know that $\dim(E_1) = 1$ and thus B will be diagonalisable if and only if $\dim(E_3) = 2$.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in E_3 \iff B \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 3 \begin{pmatrix} x \\ y \\ z \end{pmatrix} \iff \left\{ \begin{array}{rrrcl} x & -2y & -2z & = & 3x \\ -2x & -y & -4z & = & 3y \\ 2x & +4y & +7z & = & 3z \end{array} \right.$$

$$\iff \left\{ \begin{array}{rrrcl} -2x & -2y & -2z & = & 0 \\ -2x & -4y & -4z & = & 0 \\ 2x & +4y & +4z & = & 0 \end{array} \right.$$

$$\iff \left\{ \begin{array}{rcl} y & = & -z \\ x & = & 0 \end{array} \right.$$

Thus $E_3 = \text{Span} \left\{ \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}$; as $\dim(E_3) = 1 \neq m(3) = 2$ the matrix B is not diagonalisable.

Exercise 3

$$\begin{aligned}
P_A(X) &= \begin{vmatrix} 1-X & 2-2a & 1-a \\ 1 & 4-X & 1 \\ 0 & 2a-2 & a-X \end{vmatrix} \\
&\underset{L_1 \leftarrow L_1+L_3}{=} \begin{vmatrix} 1-X & 0 & 1-X \\ 1 & 4-X & 1 \\ 0 & 2a-2 & a-X \end{vmatrix} \\
&\underset{C_3 \leftarrow C_3-C_1}{=} \begin{vmatrix} 1-X & 0 & 0 \\ 1 & 4-X & 0 \\ 0 & 2a-2 & a-X \end{vmatrix} \\
&= (1-X)(4-X)(a-X)
\end{aligned}$$

Thus P_A is always split, and its roots are $1, 4$ and a.
If $a \notin \{1, 4\}$ then P_A is split with single roots, thus A is diagonalisable.
If $a \in \{1, 4\}$ then a is a double root, and A will be diagonalisable if and only if $\dim(E_a) = 2$.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in E_a \Longleftrightarrow A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = a \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \Longleftrightarrow \quad \left\{ \begin{array}{llll} x & +(2-2a)y & +(1-a)z & = ax \\ x & +4y & +z & = ay \\ & (2a-2)y & +az & = az \end{array} \right.$$

$$\Longleftrightarrow \quad \left\{ \begin{array}{llll} (1-a)x & +(2-2a)y & +(1-a)z & = 0 \\ x & +(4-a)y & +z & = 0 \\ & (2a-2)y & & = 0 \end{array} \right.$$

If $a = 1$ we get the single equation $x + 3y + z = 0$, thus $E_a = E_1 = \text{Span}\left\{ \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$; as this eigenspace is of dimension 2, A is diagonalisable.
If $a = 4$ we get $\left\{ \begin{array}{lll} y & = & 0 \\ x & = & -z \end{array} \right.$ thus $E_a = E_4 = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$, hence $\dim(E_a) = 1 \neq m(a) = 2$, and A is not diagonalisable.
Conclusion : A is diagonalisable if and only if $a \neq 4$.

Exercise 4

1.

$$\begin{aligned}
(x,y,z)\in \mathrm{Ker}(f) &\iff A\begin{pmatrix} x\\ y\\ z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\\
&\iff \left\{\begin{array}{ccccccc} 4x & + & 4y & + & 2z & = & 0\\ 4x & + & 3y & + & 3z & = & 0\\ 4x & + & 5y & + & z & = & 0\end{array}\right.\\
&\iff \left\{\begin{array}{ccccccc} 4x & + & 4y & + & 2z & = & 0\\ & - & y & + & z & = & 0\\ & & y & - & z & = & 0\end{array}\right.\\
&\iff \left\{\begin{array}{l} y=z\\ 4x+6y=0\end{array}\right.\\
&\iff \begin{pmatrix} x\\ y\\ z\end{pmatrix}=\begin{pmatrix}-\frac{3}{2}y\\ y\\ y\end{pmatrix}
\end{aligned}$$

Hence $\mathrm{Ker}(f)=\mathrm{Span}\big(\{(-3,2,2)\}\big)$.
Using the rank theorem, we deduce that $\mathrm{Im}(f)$ is of dimension 2 ; the first two columns of A form a linearly independent set of two vectors from $\mathrm{Im}(f)$ thus it is a basis : $\mathrm{Im}(f)=\mathrm{Span}\big(\{(4,4,4),(4,3,5)\}\big)$, that can be rewritten as, for instance, $\mathrm{Span}\big(\{(1,1,1),(0,-1,1)\}\big)$.

2. $\mathrm{Ker}(f)\neq\{0\}$ thus f is not injective, thus not bijective ; so, A is not invertible.
3. $\mathscr{E}$ is a set of 3 vectors from $\mathbb{R}^3$ that is of dimension 3, thus it is a basis of $\mathbb{R}^3$ if and only if it is linearly independent.

$$\begin{aligned}
\lambda(1,0,1)+\mu(2,2,2)+\nu(3,3,1)=(0,0,0) &\iff \left\{\begin{array}{ccccccc} \lambda & + & 2\mu & + & 3\nu & = & 0\\ & & 2\mu & + & 3\nu & = & 0\\ \lambda & + & 2\mu & + & \nu & = & 0\end{array}\right.\\
&\iff \left\{\begin{array}{cccl} \lambda & = & 0 & \quad (L_1-L_2)\\ \nu & = & 0 & \quad (L_1-L_3)\\ \mu & = & 0 & \end{array}\right.
\end{aligned}$$

Thus $\mathscr{E}$ is linearly independent, given its dimension it is a basis of $\mathbb{R}^3$.

4. The easiest way is to use the transfer matrices :

$$\mathrm{Mat}_{\mathscr{E}}(f)=\mathrm{Mat}_{\mathscr{B},\mathscr{E}}(id)\mathrm{Mat}_{\mathscr{B}}(f)\mathrm{Mat}_{\mathscr{E},\mathscr{B}}(id)=P^{-1}AP,$$

where P is the transfer matrix from $\mathscr{B}$ to $\mathscr{E}$:

$$P=\begin{pmatrix}1 & 2 & 3\\ 0 & 2 & 3\\ 1 & 2 & 1\end{pmatrix}$$

After calculation (using for instance a Gaussian elimination), we find that $P^{-1}=\begin{pmatrix}1 & -1 & 0\\ -\frac{3}{4} & \frac{1}{2} & \frac{3}{4}\\ \frac{1}{2} & 0 & -\frac{1}{2}\end{pmatrix}$;

hence :

$$\mathrm{Mat}_{\mathscr{E}}(f) = \begin{pmatrix} 1 & -1 & 0 \\ -\frac{3}{4} & \frac{1}{2} & \frac{3}{4} \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 4 & 4 & 2 \\ 4 & 3 & 3 \\ 4 & 5 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 2 \\ \frac{11}{4} & 10 & \frac{27}{2} \\ \frac{1}{2} & 0 & -1 \end{pmatrix}$$

Exercise 5

1.

$$\mathrm{Mat}_{\mathscr{E}}(p \circ p) = \big(\mathrm{Mat}_{\mathscr{E}}(p)\big)^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} = \mathrm{Mat}_{\mathscr{E}}(p)$$

Thus p is a projection.

2. (e_1, e_2, e_3, e_4) is a basis of E thus $\mathrm{Im}(p) = \mathrm{Span}\big(\{p(e_1), p(e_2), p(e_3), p(e_4)\}\big)$
$= \mathrm{Span}\big(\{e_1, \frac{1}{3}(e_2 + e_3 + e_4)\}\big) = \mathrm{Span}\big(\{e_1, (e_2 + e_3 + e_4)\}\big)$.

3. Using the rank theorem :

$$\dim(E) = \dim\big(\mathrm{Im}(p)\big) + \dim\big(\mathrm{Ker}(p)\big)$$

Hence $\dim\big(\mathrm{Ker}(p)\big) = \dim(E) - \dim\big(\mathrm{Im}(p)\big) = 4 - 2 = 2$.

4. We just need two find two linearly independent vectors from the kernel to get a basis of it. Here, we have $p(e_2) = p(e_3) = p(e_4)$, from this we deduce that $e_2 - e_3$ and $e_2 - e_4$ belong to $\mathrm{Ker}(p)$, and they are linearly independent as $\mathscr{E}$ is a basis of E thus linearly independent. Hence

$$\mathrm{Ker}(p) = \mathrm{Span}\big(\{e_2 - e_3, e_2 - e_4\}\big)$$

.

Exercise 6

1. $X_{n+1} = \begin{pmatrix} u_{n+3} \\ u_{n+2} \\ u_{n+1} \end{pmatrix}$; by using the relation of recurrence $u_{n+3} = -u_{n+2} + 4u_{n+1} + 4u_n$ we get

$$X_{n+1} = \begin{pmatrix} -1 & 4 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} X_n$$

If we denote this matrix M, by an obvious proof by induction $X_n = M^n X_0$.

2.

$$P_M(X) = \begin{vmatrix} -1 - X & 4 & 4 \\ 1 & -X & 0 \\ 0 & 1 & -X \end{vmatrix}$$

By developing with respect to the first row, we get :

$$\begin{aligned} P_M(X) &= (-1-X)\begin{vmatrix} -X & 0 \\ 1 & -X \end{vmatrix} - 4\begin{vmatrix} 1 & 0 \\ 0 & -X \end{vmatrix} + 4\begin{vmatrix} 1 & -X \\ 0 & 1 \end{vmatrix} \\ &= (-X-1)X^2 - 4.(-X) + 4 = -(X^3 + X^2 - 4X - 4) \\ &= -(X+1)(X^2-4) = -(X+1)(X+2)(X-2). \end{aligned}$$

$P_M(X)$ is split with single roots, thus the matrix M is diagonalisable. Let us look for a basis of each eigenspace.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in E_{-1} \Longleftrightarrow M\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\begin{pmatrix} x \\ y \\ z \end{pmatrix} \Longleftrightarrow \left\{ \begin{array}{rrrcl} -x & +4y & +4z & = & -x \\ x & & & = & -y \\ & y & & = & -z \end{array} \right.$$

$$\Longleftrightarrow \left\{ \begin{array}{rcl} y & = & -x \\ z & = & x \end{array} \right.$$

Thus $E_{-1} = \text{Span}\left\{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}\right\}$.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in E_{-2} \Longleftrightarrow M\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -2\begin{pmatrix} x \\ y \\ z \end{pmatrix} \Longleftrightarrow \left\{ \begin{array}{rrrcl} -x & +4y & +4z & = & -2x \\ x & & & = & -2y \\ & y & & = & -2z \end{array} \right.$$

$$\Longleftrightarrow \left\{ \begin{array}{rcl} x & = & -2y \\ y & = & -2z \end{array} \right.$$

Thus $E_{-2} = \text{Span}\left\{\begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix}\right\}$.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in E_{2} \Longleftrightarrow M\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2\begin{pmatrix} x \\ y \\ z \end{pmatrix} \Longleftrightarrow \left\{ \begin{array}{rrrcl} -x & +4y & +4z & = & 2x \\ x & & & = & 2y \\ & y & & = & 2z \end{array} \right.$$

$$\Longleftrightarrow \left\{ \begin{array}{rcl} x & = & 2y \\ y & = & 2z \end{array} \right.$$

Thus $E_2 = \text{Span}\left\{\begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}\right\}$.

Therefore, we can write $M = PDP^{-1}$ with $P = \begin{pmatrix} 1 & 4 & 4 \\ -1 & -2 & 2 \\ 1 & 1 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

3. We now know that $M^n = PD^nP^{-1}$. D being a diagonal matrix, $D^n = \begin{pmatrix} (-1)^n & 0 & 0 \\ 0 & (-2)^n & 0 \\ 0 & 0 & 2^n \end{pmatrix}$.

We still have to caculate P^{-1}, for instance by using a Gaussian elimination.

$$P\begin{pmatrix} x\\ y\\ z\end{pmatrix}=\begin{pmatrix} a\\ b\\ c\end{pmatrix} \iff \left\{\begin{array}{rrrcl} x & +4y & +4z & = & a\\ -x & -2y & +2z & = & b\\ x & +y & +z & = & c\end{array}\right.$$

$$\underset{\substack{L_2\leftarrow L_2+L_1\\ L_3\leftarrow L_3-L_1}}{\iff} \left\{\begin{array}{rrrcl} x & +4y & +4z & = & a\\ & 2y & +6z & = & a+b\\ & -3y & -3z & = & -a+c\end{array}\right.$$

$$\underset{L_2\leftarrow L_2+2L_3}{\iff} \left\{\begin{array}{rrrcl} x & +4y & +4z & = & a\\ & -4y & & = & -a+b+2c\\ & -3y & -3z & = & -a+c\end{array}\right.$$

$$\iff \left\{\begin{array}{rclll} a & = & -\frac{1}{3}a & & +\frac{4}{3}c\\ y & = & \frac{1}{4}a & -\frac{1}{4}b & -\frac{1}{2}c\\ z & = & \frac{1}{12}a & +\frac{1}{4}b & +\frac{1}{6}c\end{array}\right.$$

Hence $P^{-1}=\dfrac{1}{12}\begin{pmatrix} -4 & 0 & 16\\ 3 & -3 & -6\\ 1 & 3 & 2\end{pmatrix}$.

Thus, we get :

$$\begin{aligned} M^n &= \frac{1}{12}\begin{pmatrix} 1 & 4 & 4\\ -1 & -2 & 2\\ 1 & 1 & 1\end{pmatrix}\begin{pmatrix} (-1)^n & 0 & 0\\ 0 & (-2)^n & 0\\ 0 & 0 & 2^n\end{pmatrix}\begin{pmatrix} -4 & 0 & 16\\ 3 & -3 & -6\\ 1 & 3 & 2\end{pmatrix}\\ &= \frac{1}{12}\begin{pmatrix} (-1)^n & 4(-2)^n & 4.2^n\\ -(-1)^n & -2(-2)^n & 2.2^n\\ (-1)^n & (-2)^n & 2^n\end{pmatrix}\begin{pmatrix} -4 & 0 & 16\\ 3 & -3 & -6\\ 1 & 3 & 2\end{pmatrix}\\ &= \frac{1}{12}\begin{pmatrix} -4(-1)^n+12(-2)^n+4.2^n & -12(-2)^n+12.2^n & 16(-1)^n-24(-2)^n+8.2^n\\ 4(-1)^n-6(-2)^n+2.2^n & 6(-2)^n+6.2^n & -16(-1)^n+12(-2)^n+4.2^n\\ -4(-1)^n+3(-2)^n+2^n & -3(-2)^n+3.2^n & 16(-1)^n-6(-2)^n+2.2^n\end{pmatrix}\end{aligned}$$

On the last line of this marvellous equation, we read :

$$u_n=\frac{1}{12}\left(\big[-4(-1)^n+3(-2)^n+2^n\big]u_2+\big[-3(-2)^n+3.2^n\big]u_1+[16(-1)^n-6(-2)^n+2.2^n]u_0\right)$$

that is to say, using the values given in the problem :

$$u_n=\frac{1}{12}(-4(-1)^n+3(-2)^n+2^n-3(-2)^n+3.2^n)=\frac{(-1)^{n+1}+2^n}{3}$$

We can check that the first values are corresponding.