# Correction of the final exam S3

## Exercise 1 (5.5 points)

In $E=\mathbb{R}^3$, consider the family $\mathcal{F}=\big(\varepsilon_1{=}(1,-1,2),\varepsilon_2{=}(-1,4,1),\varepsilon_3{=}(1,-2,1)\big)$.

1. Is $\mathcal{F}$ a basis of $E$? If it is not, extract a maximal independent subfamily and complete it to get a basis of $E$. The final basis will be denoted by $\mathcal{B}'$.

   This family is linearly dependent because $-2\varepsilon_1+\varepsilon_2+3\varepsilon_3=0_E$. Hence, for example, $\operatorname{Span}\mathcal{F}=\operatorname{Span}(\varepsilon_1,\varepsilon_2)$ where $(\varepsilon_1,\varepsilon_2)$ is linearly independent.
   To get a basis of $E$, let us complete this family by adding the vector $\varepsilon_4=(0,0,1)$. Let us prove that $\mathcal{B}'=(\varepsilon_1,\varepsilon_2,\varepsilon_4)$ is a basis of $E$.

   - $\mathcal{B}'$ is linearly independent: for all $(a,b,c)\in\mathbb{R}^3$,

   $$\begin{aligned} a\varepsilon_1+b\varepsilon_2+c\varepsilon_4=0_E \quad &\Longrightarrow \left\{\begin{array}{lcl} a-b &=& 0\\ -a+4b &=& 0\\ 2a+b+c &=& 0\end{array}\right.\\ &\Longrightarrow \left\{\begin{array}{lcl} a-b &=& 0\\ 3b &=& 0 \quad (E_2+E_1)\\ 2a+b+c &=& 0\end{array}\right.\\ &\Longrightarrow a=b=c=0\end{aligned}$$

   - $\mathcal{B}'$ is a spanning family of $E$. Indeed,

   $$\left.\begin{array}{c}\mathcal{B}' \text{ independent}\\ \operatorname{Card}(\mathcal{B}')=3=\dim(E)\end{array}\right\}\Longrightarrow \mathcal{B}' \text{ spanning family of } E$$

     Thus,, $\mathcal{B}'$ is a basis of $E$.

2. Find the coordinates in $\mathcal{B}'$ of the vector $u=(2,0,6)$

   $u=(2,0,6)=\frac{8}{3}(1,-1,2)+\frac{2}{3}(-1,4,1)=\frac{8}{3}\varepsilon_1+\frac{2}{3}\varepsilon_2+0\varepsilon_4$.

   The coordinates of $u$ in basis $\mathcal{B}'$ are hence $\left(\frac{8}{3},\frac{2}{3},0\right)$.

3. Write the transition matrix from the standard basis $\mathcal{B}$ to basis $\mathcal{B}'$.

   The transition matrix from $\mathcal{B}$ to $\mathcal{B}'$ is $P=\begin{pmatrix}1&-1&0\\-1&4&0\\2&1&1\end{pmatrix}$.

## Exercise 2 (6.5 points)

Consider the linear map $f:\left\{\begin{array}{ccc}\mathbb{R}_2[X] & \longrightarrow & \mathbb{R}^2\\ P & \longmapsto & \left(P(1),\displaystyle\int_0^2 P(x)\,\mathrm{d}x\right)\end{array}\right.$

1. Find the matrix of $f$ in the standard bases $\left(1,X,X^2\right)$ as input basis and $\big((1,0),(0,1)\big)$ as output basis.

   If $P=1$, then $P(1)=1$ and $\displaystyle\int_0^2 P(x)\,\mathrm{d}x=\int_0^2 1\,\mathrm{d}x=2$. Hence $f(1)=(1,2)$.

   If $P=X$, then $P(1)=1$ and $\displaystyle\int_0^2 P(x)\,\mathrm{d}x=\int_0^2 x\,\mathrm{d}x=\left[\frac{x^2}{2}\right]_0^2=2$. Hence $f(X)=(1,2)$.

   If $P=X^2$, then $P(1)=1$ and $\displaystyle\int_0^2 P(x)\,\mathrm{d}x=\int_0^2 x^2\,\mathrm{d}x=\left[\frac{x^3}{3}\right]_0^2=\frac{8}{3}$. Hence $f\left(X^2\right)=\left(1,\frac{8}{3}\right)$.

   Finally, the matrix of $f$ is $A=\begin{pmatrix}1&1&1\\2&2&\frac{8}{3}\end{pmatrix}$.

2. Find a basis of $\text{Ker}(f)$ and deduce its dimension.

$$\begin{aligned}
\text{Ker}(f) &= \left\{a_0 + a_1X + a_2X^2 \in \mathbb{R}_2[X], \middle| \begin{array}{rcl} a_0 + a_1 + a_2 &=& 0 \\ 2a_0 + 2a_1 + \frac{8}{3}a_2 &=& 0 \end{array} \right\} \\
&= \left\{a_0 + a_1X + a_2X^2 \in \mathbb{R}_2[X], \middle| \begin{array}{rcll} a_0 + a_1 + a_2 &=& 0 & \\ \frac{2}{3}a_2 &=& 0 & (E_2 - 2E_1) \end{array} \right\} \\
&= \left\{a_0 + a_1X + a_2X^2 \in \mathbb{R}_2[X], a_0 = -a_1 \text{ and } a_2 = 0\right\} \\
&= \left\{a_1(X-1), a_1 \in \mathbb{R}\right\} \\
&= \text{Span}(X-1)
\end{aligned}$$

Since $(X-1)$ is linearly independent, it is a basis of $\text{Ker}(f)$. Thus, $\dim\big(\text{Ker}(f)\big) = 1$.

3. Find a basis of $\text{Im}(f)$ and deduce its dimension.

$$\text{Im}(f) = \text{Span}\underbrace{\big((1,2),(1,2),(1,\tfrac{8}{3})\big)}_{\text{dependent}} = \text{Span}\underbrace{\big((1,2),(1,\tfrac{8}{3})\big)}_{\text{independent}}$$

Thus, a basis of $\text{Im}(f)$ is $\big((1,2),(1,\frac{8}{3})\big)$ and $\dim\big(\text{Im}(f)\big) = 2$.

4. Write the rank-nullity theorem and check that your results are consistent with the theorem

Rank nullity theorem: if $f \in \mathcal{L}(E,F)$ where $E$ is finite-dimensional, then $\dim(E) = \dim\big(\text{Ker}(f)\big) + \dim\big(\text{Im}(f)\big)$.

Here, $E = \mathbb{R}_2[X]$, $\dim(E) = 3$, $\dim\big(\text{Ker}(f)\big) = 1$ and $\dim\big(\text{Im}(f)\big) = 2$. We get: $3 = 1 + 2$.

5. Find the set $S$ of all the polynomials $P \in \mathbb{R}_2[X]$ such that $f(P) = (3,8)$.

To start with, note that $P = 3X^2$ is a particular solution. Thus, for all $P \in \mathbb{R}[X]$,

$$P \in S \iff f(P) = f(3X^2) \iff f(P - 3X^2) = (0,0) \iff P - 3X^2 \in \text{Ker}(f)$$

Hence, $S = \big\{3X^2 + k(X-1), k \in \mathbb{R}\big\}$.

## Exercise 3: proving a lecture theorem (5 points)

Let $E$ be a finite-dimensional vector space, $F$ and $G$ two linear subspaces of $E$ of non-zero dimensions $n$ and $p$.

Let $\mathcal{B}_1 = (e_1, \cdots, e_n)$ be a basis of $F$ and $\mathcal{B}_2 = (\varepsilon_1, \cdots, \varepsilon_p)$ be a basis of $G$. Consider the family

$$\mathcal{F} = (e_1, \cdots, e_n, \varepsilon_1, \cdots, \varepsilon_p)$$

that is, $\mathcal{F}$ is the concatenation of $\mathcal{B}_1$ and $\mathcal{B}_2$.

Show that $F \cap G = \{0_E\} \Longrightarrow \mathcal{F}$ is linearly independent.

Let $(a_1, \cdots, a_n, b_1, \cdots, b_p) \in \mathbb{R}^{n+p}$ such that

$$a_1e_1 + \cdots + a_ne_n + b_1\varepsilon_1 + \cdots + b_p\varepsilon_p = 0_E$$

Then: $a_1e_1 + \cdots + a_ne_n = -b_1\varepsilon_1 - \cdots - b_p\varepsilon_p$.

But $\begin{cases} (e_1, \cdots, e_n) \in F^n & \Longrightarrow & a_1e_1 + \cdots + a_ne_n \in F \\ (\varepsilon_1, \cdots, \varepsilon_p) \in G^p & \Longrightarrow & -b_1\varepsilon_1 - \cdots - b_p\varepsilon_p \in G \end{cases}$

We hence deduce that: $a_1e_1 + \cdots + a_ne_n = -b_1\varepsilon_1 - \cdots - b_p\varepsilon_p \in F \cap G$.

But $F \cap G = \{0_E\}$. Thus, $a_1e_1 + \cdots + a_ne_n = -b_1\varepsilon_1 - \cdots - b_p\varepsilon_p = 0_E$.

Since $\mathcal{B}_1$ is a basis of $F$, it is independent. Thus,

$$a_1e_1 + \cdots + a_ne_n = 0_E \Longrightarrow (a_1, \cdots, a_n) = (0, \cdots, 0)$$

Similarly, $\mathcal{B}_2$ is independent and

$$-b_1\varepsilon_1 - \cdots - b_p\varepsilon_p = 0_E \Longrightarrow (-b_1, \cdots, -b_p) = (0, \cdots, 0) \Longrightarrow (b_1, \cdots, b_p) = (0, \cdots, 0)$$

Finally, $(a_1, \cdots, a_n, b_1, \cdots, b_p) = (0, \cdots, 0)$. The family $\mathcal{F}$ is hence linearly independent.

# Exercise 4: building a projector (8 points)

Let $E = \mathbb{R}^3$ together with its standard basis $\mathcal{B}$. Consider the linear subspaces

$$F = \{(x, y, z) \in E,\, x + 2y - z = 0\} \qquad \text{and} \qquad G = \left\{(x, y, z) \in E,\, \middle| \begin{array}{rcl} x + y + z & = & 0 \\ x + y - z & = & 0 \end{array} \right\}$$

1. Find a basis of $F$ and a basis of $G$.

$$\begin{array}{rcl} F & = & \{(x, y, z) \in \mathbb{R}^3,\, z = x + 2y\} \\ & = & \{(x, y, x + 2y), (x, y) \in \mathbb{R}^2\} \\ & = & \{x(1, 0, 1) + y(0, 1, 2), (x, y) \in \mathbb{R}^2\} = \text{Span}\underbrace{\big((1, 0, 1), (0, 1, 2)\big)}_{\text{independent}} \end{array}$$

A basis of $F$ is hence $\mathcal{B}_1 = \big(\varepsilon_1{=}(1, 0, 1), \varepsilon_2{=}(0, 1, 2)\big)$.

$$\begin{array}{rcl} G & = & \left\{(x, y, z) \in \mathbb{R}^3, \middle| \begin{array}{rcl} x + y + z & = & 0 \\ x + y - z & = & 0 \end{array} \right\} \\ & = & \left\{(x, y, z) \in \mathbb{R}^3, \middle| \begin{array}{rcll} x + y + z & = & 0 & \\ 2z & = & 0 & (E_1 - E_2) \end{array} \right\} \\ & = & \left\{(x, y, z) \in \mathbb{R}^3, \middle| \begin{array}{rcl} y & = & -x \\ z & = & 0 \end{array} \right\} \\ & = & \{x(1, -1, 0), x \in \mathbb{R}\} = \text{Span}\underbrace{\big((1, -1, 0)\big)}_{\text{independent}} \end{array}$$

A basis of $G$ is hence $\mathcal{B}_2 = \big(\varepsilon_3{=}(1, -1, 0)\big)$.

2. Show that $E = F \oplus G$.

It is sufficient to prove that $\mathcal{B}' = (\varepsilon_1, \varepsilon_2, \varepsilon_3)$ (concatenation of $\mathcal{B}_1$ and $\mathcal{B}_2$) is a basis of $E$.

- $\mathcal{B}'$ is linearly independent: for all $(a, b, c) \in \mathbb{R}^3$,

$$\begin{array}{rcl} a\varepsilon_1 + b\varepsilon_2 + c\varepsilon_3 = 0_E & \Longrightarrow & \left\{ \begin{array}{rcl} a + c & = & 0 \\ b - c & = & 0 \\ a + 2b & = & 0 \end{array} \right. \\ & \Longrightarrow & \left\{ \begin{array}{rcll} a + c & = & 0 & \\ a + b & = & 0 & (E_2 + E_1) \\ a + 2b & = & 0 & \end{array} \right. \\ & \Longrightarrow & \left\{ \begin{array}{rcll} a + c & = & 0 & \\ a + b & = & 0 & \\ b & = & 0 & (E_3 - E_2) \end{array} \right. \\ & \Longrightarrow & a = b = c = 0 \end{array}$$

- $\mathcal{B}'$ is a spanning family of $E$. Indeed,

$$\left. \begin{array}{c} \mathcal{B}' \text{ independent} \\ \text{Card}(\mathcal{B}') = 3 = \dim(E) \end{array} \right\} \Longrightarrow \mathcal{B}' \text{ spanning family of } E$$

Thus, $\mathcal{B}'$ is a basis of $E$, which proves that $F \oplus G = E$.

3. According to previous question, we know that for all $u \in E$, there exists a unique $(v, w) \in F \times G$ such that $u = v + w$.

Consider the endomorphism $p : u \longmapsto w$.

(a) Assume that $u \in F$. What is the value of $p(u)$? Justify.

If $u \in F$, then $u = \underbrace{u}_{\in F} + \underbrace{0_E}_{\in G} \Longrightarrow v = u$ and $w = 0_E \Longrightarrow p(u) = w = 0_E$.

(b) Assume that $u \in G$. What is the value of $p(u)$? Justify.

If $u \in G$, then $u = \underbrace{0_E}_{\in F} + \underbrace{u}_{\in G} \Longrightarrow v = 0_E$ and $w = u \Longrightarrow p(u) = w = u$.

(c) Let $\mathcal{B}'$ be the basis of $E$ resulting from the concatenation of the bases of $F$ and $G$ that you got at question 1. Find the matrix of $p$ in basis $\mathcal{B}'$ as input and output bases. This matrix is denoted by $A'$.

$\varepsilon_1 \in F \Longrightarrow p(\varepsilon_1) = 0_E$. Thus, the coordinates in $\mathcal{B}'$ of $p(\varepsilon_1)$ are $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

Similarly, $\varepsilon_2 \in F \Longrightarrow p(\varepsilon_2) = 0_E$ has coordinates $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

$\varepsilon_3 \in G \Longrightarrow p(\varepsilon_3) = \varepsilon_3 = 0\varepsilon_1 + 0\varepsilon_2 + 1\varepsilon_3$. Thus, the coordinates in $\mathcal{B}'$ of $p(\varepsilon_3)$ are $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Finally, $A' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(d) Let $A$ be the matrix of $p$ in the standard basis as input and output bases. Write the matrix relation which enables one to compute $A$. **The final computation of $A$ is not required.**

Let $P = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 2 & 0 \end{pmatrix}$ be the transition matrix from the standard basis $\mathcal{B}$ to $\mathcal{B}'$. Then:
$A' = P^{-1}AP \Longrightarrow A = PA'P^{-1}$.

## Exercise 5: solving a differential system (7 points)

We search all the differentiable real function $x$ and $y$, defined on $\mathbb{R}$ and satisfying to:

$$x(0) = 1, \quad y(0) = 2 \qquad \text{and} \qquad \forall t \in \mathbb{R}, \begin{cases} x'(t) &= -5x(t) + 4y(t) \\ y'(t) &= -6x(t) + 5y(t) \end{cases}$$

In that purpose, consider the vector function $u : \begin{cases} \mathbb{R} & \longrightarrow & \mathbb{R}^2 \\ t & \longmapsto & (x(t), y(t)) \end{cases}$ and its derivative $u' : \begin{cases} \mathbb{R} & \longrightarrow & \mathbb{R}^2 \\ t & \longmapsto & (x'(t), y'(t)) \end{cases}$.

Remind that, for all $(z_0, a) \in \mathbb{R}^2$, the unique differentiable real function $z$ satisfying to

$$z(0) = z_0 \qquad \text{and} \qquad \forall t \in \mathbb{R},\ z'(t) = az(t)$$

is the function $z : t \longmapsto z_0 e^{at}$.

1. Find $f \in \mathcal{L}(\mathbb{R}^2)$ such that for all $t \in \mathbb{R}$, $u'(t) = f\big(u(t)\big)$, and write the matrix of $f$ in the standard basis of $\mathbb{R}^2$.

   The linear map is $f : \begin{cases} \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ (x, y) & \longmapsto & (-5x + 4y, -6x + 5y) \end{cases}$

   Its matrix in the standard basis (as input and output basis) is $A = \begin{pmatrix} -5 & 4 \\ -6 & 5 \end{pmatrix}$

2. In $\mathbb{R}^2$, consider the standard basis $\mathcal{B}_1 = \big(e_1{=}(1,0), e_2{=}(0,1)\big)$ and another basis $\mathcal{B}_2 = \big(\varepsilon_1{=}(1,1), \varepsilon_2{=}(2,3)\big)$.

   (a) Write the transition matrix $P$ from $\mathcal{B}_1$ to $\mathcal{B}_2$ and find its inverse matrix $P^{-1}$.

   $$P = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} \qquad \text{and} \qquad P^{-1} = \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}$$

   (b) Find the coordinates of $u(0) = (1, 2)$ in basis $\mathcal{B}_2$.

   The coordinates of $u(0)$ in $\mathcal{B}_2$ are $X_2(0) = P^{-1}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

   We can check that $u(0) = -\varepsilon_1 + \varepsilon_2$.

(c) Find the matrix of $f$ in basis $\mathcal{B}_2$ as input and output bases.

$f(\varepsilon_1) = (-1,-1) = -\varepsilon_1 + 0\varepsilon_2$, whose coordinates in $\mathcal{B}_2$ are $\begin{pmatrix}-1\\0\end{pmatrix}$.

$f(\varepsilon_2) = (2,3) = 0\varepsilon_1 + \varepsilon_2$, whose coordinates in $\mathcal{B}_2$ are $\begin{pmatrix}0\\1\end{pmatrix}$.

The matrix of $f$ in $\mathcal{B}_2$ as input and output basis is hence $D = \begin{pmatrix}-1 & 0\\0 & 1\end{pmatrix}$.

(d) Let $X_2(t) = \begin{pmatrix}x_2(t)\\y_2(t)\end{pmatrix}$ and $X_2'(t) = \begin{pmatrix}x_2'(t)\\y_2'(t)\end{pmatrix}$ be the columns containing the coordinates in $\mathcal{B}_2$ of the vectors $u(t)$ and $u'(t)$. Find a matrix relation which enables one to get $X_2'(t)$ depending on $X_2(t)$.

$$\forall t \in \mathbb{R}, u'(t) = f\big(u(t)\big) \Longrightarrow X_2'(t) = DX_2(t) \Longrightarrow \begin{cases} x_2'(t) &= -x_2(t) \\ y_2'(t) &= y_2(t) \end{cases}$$

(e) Find the functions $t \longmapsto x_2(t)$ and $t \longmapsto y_2(t)$.

Using previous questions, we get:

$$x_2(0) = -1 \quad \text{and} \quad \forall t \in \mathbb{R}, x_2'(t) = -x_2(t) \Longrightarrow x_2(t) = -e^{-t}$$

Similarly,

$$y_2(0) = 1 \quad \text{and} \quad \forall t \in \mathbb{R}, y_2'(t) = y_2(t) \Longrightarrow y_2(t) = e^{t}$$

(f) Deduce the functions $t \longmapsto x(t)$ and $t \longmapsto y(t)$.

For all $t \in \mathbb{R}$, the column $X(t) = \begin{pmatrix}x(t)\\y(t)\end{pmatrix}$ contains the coordinates of $u(t)$ in the standard basis. Furthermore, we know that $X(t) = PX_2(t)$. Hence,

$$\begin{pmatrix}x(t)\\y(t)\end{pmatrix} = \begin{pmatrix}1 & 2\\1 & 3\end{pmatrix}\begin{pmatrix}-e^{-t}\\e^{t}\end{pmatrix} = \begin{pmatrix}-e^{-t}+2e^{t}\\-e^{-t}+3e^{t}\end{pmatrix}$$

Finally, for all $t \in \mathbb{R}$,

$$x(t) = -e^{-t} + 2e^{t} \qquad \text{and} \qquad y(t) = -e^{-t} + 3e^{t}$$

# Exercise 6: diagonalization of square matrices (8 points)

Consider the matrices $A = \begin{pmatrix}2 & -4 & 1\\0 & -2 & 1\\4 & -5 & 0\end{pmatrix}$ and $B = \begin{pmatrix}2 & 5 & 5\\-5 & -8 & -5\\5 & 5 & 2\end{pmatrix}$.

1. Compute **in factorized form** their characteristic polynomials. Check that the eigenvalues of $A$ are $-1$ and 2, and that the eigenvalues of $B$ are $-3$ and 2.

$$\begin{aligned}
P_A(X) &= \begin{vmatrix}2-X & -4 & 1\\0 & -2-X & 1\\4 & -5 & -X\end{vmatrix} = \begin{vmatrix}2-X & -2+X & 0\\0 & -2-X & 1\\4 & -5 & -X\end{vmatrix} \quad (R_1 \leftarrow R_1 - R_2)\\
&= \begin{vmatrix}2-X & 0 & 0\\0 & -2-X & 1\\4 & -1 & -X\end{vmatrix} \quad (C_2 \leftarrow C_2 + C_1)\\
&= (2-X) \times \big[(-2-X)(-X)+1\big]\\
&= (2-X) \times \underbrace{\big[X^2+2X+1\big]}_{(X+1)^2}
\end{aligned}$$

$$P_B(X) = \begin{vmatrix} 2-X & 5 & 5 \\ -5 & -8-X & -5 \\ 5 & 5 & 2-X \end{vmatrix} = \begin{vmatrix} -3-X & 5 & 5 \\ 3+X & -8-X & -5 \\ 0 & 5 & 2-X \end{vmatrix} \quad (C_1 \leftarrow C_1 - C_2)$$

$$= \begin{vmatrix} -3-X & 5 & 5 \\ 0 & -3-X & 0 \\ 0 & 5 & 2-X \end{vmatrix} \quad (R_2 \leftarrow R_2 + R_1)$$

$$= (-3-X) \times \big[(-3-X)(2-X) - 0\big] = (-3-X)^2(2-X)$$

2. Are the matrices $A$ and $B$ diagonalizable in $\mathscr{M}_3(\mathbb{R})$? If they are, find the matrices $P$ and $D$.
   Be accurate in your redaction.

   - Matrix $A$:
     $\mathrm{Sp}(A) = \{-1, 2\}$ with $m(-1) = 2$ and $m(2) = 1$. Hence, $A$ is diagonalizable iff $\dim(E_{-1}) = 2$.

$$E_{-1} = \left\{ (x,y,z) \in \mathbb{R}^3, \left| \begin{array}{rcl} 3x - 4y + z & = & 0 \\ -y + z & = & 0 \\ 4x - 5y + z & = & 0 \end{array} \right. \right\}$$

$$= \left\{ (x,y,z) \in \mathbb{R}^3, \left| \begin{array}{rcll} z & = & y & (E_2) \\ 4x & = & 4y & (E_3) \\ 0y & = & 0 & (E_1) \end{array} \right. \right\}$$

$$= \{y(1,1,1), y \in \mathbb{R}\}$$

$$= \mathrm{Span}\,\big((1,1,1)\big)$$

     Thus, $\dim(E_{-1}) = 1 \neq m(-1)$ and $A$ is not diagonalizable.

   - Matrix $B$:
     $\mathrm{Sp}(B) = \{-3, 2\}$ with $m(-3) = 2$ and $m(2) = 1$. Hence, $B$ is diagonalizable iff $\dim(E_{-3}) = 2$.

$$E_{-3} = \left\{ (x,y,z) \in \mathbb{R}^3, \left| \begin{array}{rcl} 5x + 5y + 5z & = & 0 \\ -5x - 5y - 5z & = & 0 \\ 5x + 5y + 5z & = & 0 \end{array} \right. \right\}$$

$$= \big\{(x,y,z) \in \mathbb{R}^3, z = -x - y\big\}$$

$$= \{x(1,0,-1) + y(0,1,-1), (x,y) \in \mathbb{R}^2\}$$

$$= \mathrm{Span}\,\underbrace{\big((1,0,-1), (0,1,-1)\big)}_{\text{independent}}$$

     Thus, $\dim(E_{-3}) = 2 = m(-3)$ and $B$ is diagonalizable.
     Furthermore,

$$E_2 = \left\{ (x,y,z) \in \mathbb{R}^3, \left| \begin{array}{rcl} 5y + 5z & = & 0 \\ -5x - 10y - 5z & = & 0 \\ 5x + 5y & = & 0 \end{array} \right. \right\}$$

$$= \left\{ (x,y,z) \in \mathbb{R}^3, \left| \begin{array}{rcll} z & = & -y & \\ x & = & -y & (E_3) \\ 0y & = & 0 & (E_2) \end{array} \right. \right\}$$

$$= \{y(-1,1,-1), y \in \mathbb{R}\}$$

$$= \mathrm{Span}\,\big((-1,1,-1)\big)$$

     Thus, $P = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & -1 & -1 \end{pmatrix}$ and $D = \begin{pmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.