EPITA

Mathematics

Final exam S3

December 2022

Duration: 3 hours

Name:

First name:

Class:

MARK:

The marking system is for a mark between 0 and 40. It will be divided by 2, to get a mark between 0 and 20.

Instructions:

- Documents and pocket calculators are not allowed.
- Write your answers on the stapled sheets provided for answering. No other sheet will be corrected.
- Please, do not use lead pencils for answering.

Exercise 1 (6 points)

An internet service provider has an hotline service, in order to assist the customers having connection problems. For a 1-hour time interval, consider the random variable

$$X = \text{"Number of calls to the hotline service during this 1-hour time interval"}$$

Assume that the numbers of calls, in two non-overlapping time intervals, are independent random variables. We accept without proof that, in this hypothesis, there exists $\lambda > 0$ such that $X \rightsquigarrow \text{Poisson}(\lambda)$, that is,

$$X(\Omega) = \mathbb{N} \qquad \text{and} \qquad \forall n \in \mathbb{N},\ P(X{=}n) = e^{-\lambda}\frac{\lambda^n}{n!}$$

The hotline service is opened 10 hours each day (from 9:00 to 19:00), and the value of λ is the same for all 1-hour time interval contained in the opening hours.

1. Find the generating function $G_X(t)$ of variable X. First, express $G_X(t)$ as a power series, then express it with the usual functions.

2. Compute the expectation and the variance of X.

3. Consider a day d and the random variable

$$Y = \text{"Number of calls to the hotline service during the whole day"}$$

   (a) Find the generating function G_Y of variable Y. Justify accurately.

   (b) Deduce the distribution of Y.

Exercise 2 (6.5 points)

Consider the linear map $f : \left\{ \begin{array}{ccc} \mathbb{R}_2[X] & \longrightarrow & \mathbb{R}^2 \\ P & \longmapsto & \big(P(1), P(2)\big) \end{array} \right.$

1. Let $P = aX^2 + bX + c \in \mathbb{R}_2[X]$. Write the conditions on (a, b, c) for $P \in \mathrm{Ker}(f)$. Then find a basis of $\mathrm{Ker}(f)$.

2. Find the rank of f, then $\mathrm{Im}(f)$.

3. In $\mathbb{R}_2[X]$, consider the polynomials $P_1 = -X + 2$ and $P_2 = X - 1$. Compute $P_i(1)$ and $P_i(2)$ for $i \in \{1, 2\}$.

4. Find a basis $\mathcal{B}$ of $\mathbb{R}_2[X]$ such that the matrix of f in this basis $\mathcal{B}$ as input basis and in the standard output basis is $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

5. Find the set S of all the polynomials $P \in \mathbb{R}_2[X]$ such that $f(P) = (42, 1)$.

Exercise 3 (8 points)

Consider the matrices $A = \begin{pmatrix} -1 & -1 & -2 \\ 2 & 2 & 2 \\ 2 & 1 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} -4 & -2 & 4 \\ -6 & -5 & 8 \\ -6 & -4 & 7 \end{pmatrix}$.

1. Compute **in factorized form** the characteristic polynomials of A and B. Check that the eigenvalues of A are 1 and 2, and that the eigenvalues of B are 0 and -1.

2. Are matrices A and B diagonalizable in $\mathscr{M}_3(\mathbb{R})$? If they are, find P and D.
   Be accurate in your redaction.

Exercise 4: a theorem's proof (6.5 points)

Let E be a finite-dimensional vector space, F and G two linear subspaces of E of non-zero dimensions n and p.

Consider $\mathcal{B}_1 = (e_1, \cdots, e_n)$ a basis of F and $\mathcal{B}_2 = (\varepsilon_1, \cdots, \varepsilon_p)$ a basis of G.

Assume that the concatenated family $\mathcal{B} = (e_1, \cdots, e_n, \varepsilon_1, \cdots, \varepsilon_p)$ is a basis of E.

1. What can be said about F and G in this case?

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2. Prove this property.

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Exercise 5: building a symmetry (8 points)

Let us work in the vector space $E = \mathbb{R}^3$ and its standard basis $\mathcal{B}$. Consider the linear subspaces

$$F = \{(x, y, z) \in E,\ x - y + 2z = 0\} \qquad \text{and} \qquad G = \left\{ (x, y, z) \in E, \left| \begin{array}{rcl} x + y + z & = & 0 \\ x - y + z & = & 0 \end{array} \right. \right\}$$

1. Find a basis of F and a basis of G.

2. Show that $E = F \oplus G$.

3. According to the previous question, we know that for all $u \in E$, there exists a unique $(v, w) \in F \times G$ such that $u = v + w$.

   Consider the endomorphism $s : u \longmapsto v - w$.

   (a) Assume that $u \in F$. What is the value of $s(u)$?

   (b) Assume that $u \in G$. What is the value of $s(u)$?

(c) Let $\mathcal{B}'$ be the concatenation of the bases of F and G that you got at question 1. We know that it is a basis of E. What is the matrix of s in basis $\mathcal{B}'$ as input and output basis. This matrix is denoted by A'.

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(d) Let A be the matrix of s in the standard basis as input and output basis. Write the formula which enables one to compute A. **We don't ask you to do the computation.**

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Exercise 6: Probabilities (5 points)

Let $p \in]0,1[$. Consider a random variable X which is geometric-distributed with parameter p.

1. Write explicitly the distribution of X.

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2. Let $(k,n) \in (\mathbb{N}^*)^2$.

(a) Show that $P(X{>}n) = q^n$ where $q = 1-p$.

Hint: you can start by writing $P(X{>}n) = \displaystyle\sum_{k=n+1}^{+\infty} P(X{=}k)$ or, alternatively, $P(X{>}n) = 1 - \displaystyle\sum_{k=1}^{n} P(X{=}k)$.

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(b) Explain why $P(X{=}n{+}k \ \cap \ X{>}n) = P(X{=}n{+}k)$.

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(c) Compute the conditional probability $P(X{=}n{+}k \mid X{>}n)$. Compare your result with the value of $P(X{=}k)$.

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(d) Explain why we say that the distribution of X is "memoryless".

3. Consider a random variable Y such that

$$Y(\Omega) = \mathbb{N}^* \qquad \text{and} \qquad \forall (k,n) \in (\mathbb{N}^*)^2\,,\ P(Y{=}n{+}k \mid Y{>}n) = P(Y{=}k)$$

Let (p_n) be the sequence defined for all $n \in \mathbb{N}^*$ by: $p_n = P(Y{=}n)$.

(a) Express $P(Y{>}1)$ as a function of p_1.

(b) By using the events "$Y{>}1$", "$Y{=}1$" and "$Y{=}2$", express $\dfrac{p_2}{p_1}$ as a function of p_1.

(c) Similarly, for all $n \in \mathbb{N}^*$, by using the events "$Y{>}1$", "$Y{=}n$" and "$Y{=}n+1$", find a simple expression of $\dfrac{p_{n+1}}{p_n}$.

(d) Deduce the value of p_n as a function of n. How do we call the distribution of Y?