Correction of the final exam

Exercise 1 (6 points)

An internet service provider has an hotline service, in order to assist the customers having connection problems. For a 1-hour time interval, consider the random variable

$$X = \text{"Number of calls to the hotline service during this 1-hour time interval"}$$

Assume that the numbers of calls, in two non-overlapping time intervals, are independent random variables. We accept without proof that, in this hypothesis, there exists $\lambda > 0$ such that $X \rightsquigarrow \text{Poisson}(\lambda)$, that is,

$$X(\Omega) = \mathbb{N} \qquad \text{and} \qquad \forall n \in \mathbb{N},\ P(X{=}n) = e^{-\lambda}\frac{\lambda^n}{n!}$$

The hotline service is opened 10 hours each day (from 9:00 to 19:00), and the value of λ is the same for all 1-hour time interval contained in the opening hours.

1. Find the generating function $G_X(t)$ of variable X. First, express $G_X(t)$ as a power series, then express it with the usual functions.

   $$G_X(t) = \sum_{n=0}^{+\infty} P(X{=}n)t^n = \sum_{n=0}^{+\infty} e^{-\lambda}\frac{\lambda^n}{n!}t^n = e^{-\lambda}\sum_{n=0}^{+\infty}\frac{(\lambda t)^n}{n!}$$

   Hence, $G_X(t) = e^{-\lambda}\, e^{\lambda t} = e^{\lambda(t-1)}$.

2. Compute the expectation and the variance of X.

   $G'_X(t) = \lambda e^{\lambda(t-1)}$. Thus, $\mathrm{E}(X) = G'_X(1) = \lambda$

   $G''_X(t) = \lambda^2 e^{\lambda(t-1)}$. Thus, $\mathrm{Var}(X) = G''_X(1) + \mathrm{E}(X) - \mathrm{E}^2(X) = \lambda^2 + \lambda - \lambda^2 = \lambda$.

3. Consider a day d and the random variable

   $$Y = \text{"Number of calls to the hotline service during the whole day"}$$

   (a) Find the generating function G_Y of variable Y. Justify accurately.

   We can split the opening hours of the service into ten 1-hour periods. For $h \in [\![1, 10]\!]$, let us define

   $$X_h = \text{"Number of calls to the hotline service during period } h\text{"}$$

   Then $Y = X_1 + X_2 + \cdots + X_{10}$ and the variables X_h are independent since the 1-hour periods don't overlap.

   Thus, $G_Y(t) = G_{X_1}(t) \times \cdots \times G_{X_{10}}(t) = e^{10\lambda(t-1)}$.

   (b) Deduce the distribution of Y.

   We deduce from question 3.a that Y is Poisson-distributed with parameter 10λ:

   $$Y(\Omega) = \mathbb{N} \qquad \text{and} \qquad \forall n \in \mathbb{N},\ P(Y{=}n) = e^{-10\lambda}\frac{(10\lambda)^n}{n!}$$

Exercise 2 (6.5 points)

Consider the linear map $f : \left\{\begin{array}{ccc} \mathbb{R}_2[X] & \longrightarrow & \mathbb{R}^2 \\ P & \longmapsto & \big(P(1), P(2)\big) \end{array}\right.$

1. Let $P = aX^2 + bX + c \in \mathbb{R}_2[X]$. Write the conditions on (a, b, c) for $P \in \operatorname{Ker}(f)$. Then find a basis of $\operatorname{Ker}(f)$.

$$P \in \operatorname{Ker}(f) \iff \begin{cases} a+b+c &= 0 \quad (P(1)=0) \\ 4a+2b+c &= 0 \quad (P(2)=0) \end{cases} \iff \begin{cases} a+b+c &= 0 \\ 3a+b &= 0 \quad (\text{Eq}_2 - \text{Eq}_1) \end{cases} \iff \begin{cases} b &= -3a \\ c &= -a-b = 2a \end{cases}$$

Thus, $\operatorname{Ker}(f) = \{a(X^2 - 3X + 2), a \in \mathbb{R}\} = \operatorname{Span}(X^2 - 3X + 2)$.
Since the family $(X^2 - 3X + 2)$ is linearly independent, it is a basis of $\operatorname{Ker}(f)$.

2. Find the rank of f, then $\operatorname{Im}(f)$.

According to the rank-nullity theorem: $\dim(\mathbb{R}_2[X]) = \dim(\operatorname{Ker}(f)) + \dim(\operatorname{Im}(f))$.

Thus, $\operatorname{rank}(f) = \dim(\operatorname{Im}(f)) = 3 - 1 = 2$.

We hence get: $\operatorname{Im}(f) \subset \mathbb{R}^2$ and $\dim(\operatorname{Im}(f)) = \dim(\mathbb{R}^2) \Longrightarrow \operatorname{Im}(f) = \mathbb{R}^2$.

3. In $\mathbb{R}_2[X]$, consider the polynomials $P_1 = -X + 2$ and $P_2 = X - 1$. Compute $P_i(1)$ and $P_i(2)$ for $i \in \{1, 2\}$.

$P_1(1) = 1$, $P_1(2) = 0$ and $P_2(1) = 0$, $P_2(2) = 1$.

4. Find a basis $\mathcal{B}$ of $\mathbb{R}_2[X]$ such that the matrix of f in this basis $\mathcal{B}$ as input basis and in the standard output basis is $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

Consider the family $\mathcal{B} = (P_1, P_2, P_3{=}X^2 - 3X + 2)$. This family is a basis of $\mathbb{R}_2[X]$. Furthermore, using questions 1 and 3,

$$f(P_1) = (1, 0), \quad f(P_2) = (0, 1) \quad \text{and} \quad f(P_3) = (0, 0)$$

The matrix of f in this basis $\mathcal{B}$ as input and the standard basis as output is hence the required A matrix.

5. Find the set S of all the polynomials $P \in \mathbb{R}_2[X]$ such that $f(P) = (42, 1)$.

Let us express P in basis $\mathcal{B}$: $P = aP_1 + bP_2 + cP_3$ where $(a, b, c) \in \mathbb{R}^3$.

The coordinates of $f(P)$ in the standard basis of $\mathbb{R}^2$ are given by $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$

Thus,, $f(P) = (a, b)$. It results that:

$$S = \{aP_1 + bP_2 + cP_3 \text{ such that } a = 42 \text{ and } b = 1\} = \{42P_1 + P_2 + cP_3,\ c \in \mathbb{R}\} = \{-41X + 83 + cP_3,\ c \in \mathbb{R}\}$$

Exercise 3 (8 points)

Consider the matrices $A = \begin{pmatrix} -1 & -1 & -2 \\ 2 & 2 & 2 \\ 2 & 1 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} -4 & -2 & 4 \\ -6 & -5 & 8 \\ -6 & -4 & 7 \end{pmatrix}$.

1. Compute **in factorized form** the characteristic polynomials of A and B. Check that the eigenvalues of A are 1 and 2, and that the eigenvalues of B are 0 and -1.

$$P_A[X] = \begin{vmatrix} -1-X & -1 & -2 \\ 2 & 2-X & 2 \\ 2 & 1 & 3-X \end{vmatrix} = \begin{vmatrix} 1-X & -1 & -2 \\ 0 & 2-X & 2 \\ -1+X & 1 & 3-X \end{vmatrix} \quad (C_1 \leftarrow C_1 - C_3)$$

$$= \begin{vmatrix} 1-X & -1 & -2 \\ 0 & 2-X & 2 \\ 0 & 0 & 1-X \end{vmatrix} \quad (R_3 \leftarrow R_3 + R_1)$$

$$= (1-X)^2(2-X)$$

Furthermore,

$$P_B[X] = \begin{vmatrix} -4-X & -2 & 4 \\ -6 & -5-X & 8 \\ -6 & -4 & 7-X \end{vmatrix} = \begin{vmatrix} -4-X & -2 & 4 \\ -6 & -5-X & 8 \\ 0 & 1+X & -1-X \end{vmatrix} \quad (R_3 \leftarrow R_3 - R_2)$$

$$= \begin{vmatrix} -4-X & 2 & 4 \\ -6 & 3-X & 8 \\ 0 & 0 & -1-X \end{vmatrix} \quad (C_2 \leftarrow C_2 + C_3)$$

$$= (-1-X)\begin{vmatrix} -4-X & 2 \\ -6 & 3-X \end{vmatrix}$$

Hence, $P_B(X) = (-1-X)\big[(-4-X)(3-X)+12\big] = -(1+X)\big[X^2+X\big] = -X(X+1)^2$.

2. Are matrices A and B diagonalizable in $\mathscr{M}_3(\mathbb{R})$? If they are, find P and D.
Be accurate in your redaction.

Matrix A: P_A is split, $\mathrm{Sp}(A) = \{1,2\}$ with $m(1)=2$ and $m(2)=1$. Thus, A is diagonalizable if and only if $\dim(E_1)=2$.

$$\begin{aligned} E_1 &= \left\{(x,y,z)\in\mathbb{R}^3, \left| \begin{array}{rcl} -2x-y-2z &=& 0 \\ 2x+y+2z &=& 0 \\ 2x+y+2z &=& 0 \end{array}\right.\right\} \\ &= \big\{(x,y,z)\in\mathbb{R}^3,\ y=-2x-2z\big\} \\ &= \big\{(x,-2x-2z,z),\ (x,z)\in\mathbb{R}^2\big\} \\ &= \big\{x(1,-2,0)+z(0,-2,1),\ (x,z)\in\mathbb{R}^2\big\} = \mathrm{Span}\big((1,-2,0),(0,-2,1)\big) \end{aligned}$$

Hence, $\dim(E_1)=2$ and A is diagonalizable.

$$\begin{aligned} E_2 &= \left\{(x,y,z)\in\mathbb{R}^3, \left| \begin{array}{rcl} -3x-y-2z &=& 0 \\ 2x+2z &=& 0 \\ 2x+y+z &=& 0 \end{array}\right.\right\} \\ &= \left\{(x,y,z)\in\mathbb{R}^3, \left| \begin{array}{rcl} -3x-y-2z &=& 0 \\ 2x+2z &=& 0 \\ -x-z &=& 0 \quad (\mathrm{Eq}_3 \leftarrow \mathrm{Eq}_3+\mathrm{Eq}_1) \end{array}\right.\right\} \\ &= \left\{(x,y,z)\in\mathbb{R}^3, \left| \begin{array}{rcl} z &=& -x \\ y &=& -3x-2z=-x \end{array}\right.\right\} \\ &= \big\{x(1,-1,-1),\ x\in\mathbb{R}\big\} = \mathrm{Span}\big((1,-1,-1)\big) \end{aligned}$$

Finally, $P = \begin{pmatrix} 1 & 0 & 1 \\ -2 & -2 & -1 \\ 0 & 1 & -1 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

Matrix B: P_B is split, $\mathrm{Sp}(A) = \{0,-1\}$ with $m(-1)=2$ and $m(0)=1$. Thus, B is diagonalizable if and only if $\dim(E_{-1})=2$.

$$\begin{aligned} E_{-1} &= \left\{(x,y,z)\in\mathbb{R}^3, \left| \begin{array}{rcl} -3x-2y+4z &=& 0 \\ -6x-4y+8z &=& 0 \\ -6x-4y+8z &=& 0 \end{array}\right.\right\} \\ &= \big\{(x,y,z)\in\mathbb{R}^3,\ y=-\tfrac{3}{2}x+2z\big\} \\ &= \big\{(x,-\tfrac{3}{2}x+2z,z),\ (x,z)\in\mathbb{R}^2\big\} \\ &= \big\{x(1,-3/2,0)+z(0,2,1),\ (x,z)\in\mathbb{R}^2\big\} = \mathrm{Span}\big((1,-3/2,0),(0,2,1)\big) \end{aligned}$$

Hence, $\dim(E_{-1})=2$ and B is diagonalizable.

$$E_0 = \left\{(x,y,z)\in\mathbb{R}^3,\ \left|\begin{array}{rcl} -4x-2y+4z &=& 0\\ -6x-5y+8z &=& 0\\ -6x-4y+7z &=& 0\end{array}\right.\right\}$$

$$= \left\{(x,y,z)\in\mathbb{R}^3,\ \left|\begin{array}{rcll} y-z &=& 0 & (\mathrm{Eq}_1 \leftarrow \tfrac{3}{2}\mathrm{Eq}_1 - \mathrm{Eq}_3)\\ y-z &=& 0 & (\mathrm{Eq}_2 \leftarrow \mathrm{Eq}_3 - \mathrm{Eq}_2)\\ -6x-4y+7z &=& 0 & \end{array}\right.\right\}$$

$$= \left\{(x,y,z)\in\mathbb{R}^3,\ \left|\begin{array}{rcl} y &=& z\\ 6x &=& 3z\end{array}\right.\right\}$$

$$= \big\{x(1,2,2),\ x\in\mathbb{R}\big\} = \mathrm{Span}\big((1,2,2)\big)$$

Finally, $P = \begin{pmatrix} 1 & 0 & 1\\ -3/2 & 2 & 2\\ 0 & 1 & 2\end{pmatrix}$ and $D = \begin{pmatrix} -1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 0\end{pmatrix}$.

Exercise 4: a theorem's proof (6.5 points)

Let E be a finite-dimensional vector space, F and G two linear subspaces of E of non-zero dimensions n and p.

Consider $\mathcal{B}_1 = (e_1, \cdots, e_n)$ a basis of F and $\mathcal{B}_2 = (\varepsilon_1, \cdots, \varepsilon_p)$ a basis of G.

Assume that the concatenated family $\mathcal{B} = (e_1, \cdots, e_n, \varepsilon_1, \cdots, \varepsilon_p)$ is a basis of E.

1. What can be said about F and G in this case?

   They are supplementary in E: $E = F \oplus G$.

2. Prove this property.

   Assume that $\mathcal{B} = (e_1, \cdots, e_n, \varepsilon_1, \cdots, \varepsilon_p)$ is a basis of E. Let us show that $E = F \oplus G$.

   (a) $F \cap G = \{0_E\}$.

   To start with, it is evident that $\{0_E\} \subset F \cap G$. Indeed, since F and G are linear subspaces of E, they both contain the zero vector.

   Let us show that $F \cap G \subset \{0_E\}$: let $u \in F \cap G$. Then:

$$\left.\begin{array}{c} u \in F\\ F = \mathrm{Span}(\mathcal{B}_1)\end{array}\right\} \Longrightarrow \exists (a_1, \cdots, a_n) \in \mathbb{R}^n, u = a_1e_1 + \cdots + a_ne_n$$

   Similarly,

$$\left.\begin{array}{c} u \in G\\ G = \mathrm{Span}(\mathcal{B}_2)\end{array}\right\} \Longrightarrow \exists (b_1, \cdots, b_p) \in \mathbb{R}^p, u = b_1\varepsilon_1 + \cdots + b_p\varepsilon_p$$

   By substracting these two expressions of u, we get:

$$0_E = a_1e_1 + \cdots + a_ne_n - b_1\varepsilon_1 - \cdots - b_p\varepsilon_p$$

   Since the family $\mathcal{B}$ is independent, we deduce that $(a_1, \cdots, a_n, -b_1, \cdots, -b_p) = (0, \cdots, 0)$.

   This leads to: $u = a_1e_1 + \cdots + a_ne_n = 0_E$.

   (b) $E = F + G$.

   To start with, it is evident that $F + G \subset E$: since F and G are linear subspaces of E, $F + G$ is a linear subspace of E too.

   Let us prove that $E \subset F + G$. Let $u \in E$ and let's show that $u \in F + G$.

   The family $\mathcal{B}$ is a basis of E. It is hence a spanning family. Thus, there exists $(a_1, \cdots, a_n, b_1, \cdots, b_p) \in \mathbb{R}^{n+p}$ such that

$$u = a_1e_1 + \cdots + a_ne_n + b_1\varepsilon_1 + \cdots + b_p\varepsilon_p$$

Then

$$\left\{\begin{array}{lll}(e_1,\cdots,e_n)\in F^n & \Longrightarrow & a_1e_1+\cdots+a_ne_n\in F\\ (\varepsilon_1,\cdots,\varepsilon_p)\in G^p & \Longrightarrow & b_1\varepsilon_1+\cdots+b_p\varepsilon_p\in G\end{array}\right.$$

Finally,

$$u=\underbrace{a_1e_1+\cdots+a_ne_n}_{\in F}+\underbrace{b_1\varepsilon_1+\cdots+b_p\varepsilon_p}_{\in G}\Longrightarrow u\in F+G$$

Exercise 5: building a symmetry (8 points)

Let us work in the vector space $E=\mathbb{R}^3$ and its standard basis $\mathcal{B}$. Consider the linear subspaces

$$F=\{(x,y,z)\in E,\ x-y+2z=0\}\qquad\text{and}\qquad G=\left\{(x,y,z)\in E,\ \left|\begin{array}{lll}x+y+z&=&0\\x-y+z&=&0\end{array}\right.\right\}$$

1. Find a basis of F and a basis of G.

$$\begin{aligned}F&=\left\{(x,y,z)\in\mathbb{R}^3,\ y=x+2z\right\}\\&=\left\{(x,x+2z,z),\ (x,z)\in\mathbb{R}^2\right\}\\&=\left\{x(1,1,0)+z(0,2,1),\ (x,z)\in\mathbb{R}^2\right\}=\operatorname{Span}\big((1,1,0),(0,2,1)\big)\end{aligned}$$

The family $\big((1,1,0),(0,2,1)\big)$ is also independent, it is hence a basis of F.

$$\begin{aligned}G&=\left\{(x,y,z)\in\mathbb{R}^3,\ \left|\begin{array}{lll}x+y+z&=&0\\x-y+z&=&0\end{array}\right.\right\}\\&=\left\{(x,y,z)\in\mathbb{R}^3,\ \left|\begin{array}{llll}x+y+z&=&0&\\2y&=&0&(\text{Eq}_2\leftarrow\text{Eq}_1-\text{Eq}_2)\end{array}\right.\right\}\\&=\left\{(x,y,z)\in\mathbb{R}^3,\ \left|\begin{array}{lll}z&=&-x\\y&=&0\end{array}\right.\right\}\\&=\left\{x(1,0,-1),\ x\in\mathbb{R}\right\}=\operatorname{Span}\big((1,0,-1)\big)\end{aligned}$$

The family $\big((1,0,-1)\big)$ is also independent, it is hence a basis of G.

2. Show that $E=F\oplus G$.

It is sufficient to show that the family $\mathcal{F}=\big(\varepsilon_1{=}(1,1,0),\varepsilon_2{=}(0,2,1),\varepsilon_3{=}(1,0,-1)\big)$ (the concatenation of the bases of F and G) is a basis E.

<u>$\mathcal{F}$ is independent:</u> for all $(a,b,c)\in\mathbb{R}^3$,

$$a\varepsilon_1+b\varepsilon_2+c\varepsilon_3=0_E\Longrightarrow\left\{\begin{array}{lll}a+c&=&0\\a+2b&=&0\\b-c&=&0\end{array}\right.\Longrightarrow\left\{\begin{array}{llll}a+c&=&0&\\2b-c&=&0&(\text{Eq}_2-\text{Eq}_1)\\b-c&=&0&\end{array}\right.$$

$$\Longrightarrow\left\{\begin{array}{llll}a+c&=&0&\\2b-c&=&0&\\-b&=&0&(\text{Eq}_3-\text{Eq}_2)\end{array}\right.\Longrightarrow a=b=c=0$$

<u>$\mathcal{F}$ is a spanning family:</u>

$$\left.\begin{array}{c}\mathcal{F}\text{ is independent}\\\operatorname{Card}(\mathcal{F})=\dim(E)\end{array}\right\}\Longrightarrow\operatorname{Span}\mathcal{F}=E$$

Thus, $E=F\oplus G$.

3. According to the previous question, we know that for all $u\in E$, there exists a unique $(v,w)\in F\times G$ such that $u=v+w$.

Consider the endomorphism $s:u\longmapsto v-w$.

(a) Assume that $u \in F$. What is the value of $s(u)$?

If $u \in F$, then $u = \underbrace{u}_{\in F} + \underbrace{0_E}_{\in G} \Longrightarrow v = u$ and $w = 0_E$.

Hence, $s(u) = v - w = u - 0_E = u$.

(b) Assume that $u \in G$. What is the value of $s(u)$?

If $u \in G$, then $u = \underbrace{0_E}_{\in F} + \underbrace{u}_{\in G} \Longrightarrow v = 0_E$ and $w = u$.

Hence $s(u) = v - w = 0_E - u = -u$.

(c) Let $\mathcal{B}'$ be the concatenation of the bases of F and G that you got at question 1. We know that it is a basis of E. What is the matrix of s in basis $\mathcal{B}'$ as input and output basis. This matrix is denoted by A'.

The basis $\mathcal{B}'$ is the family $\mathcal{F} = \big(\varepsilon_1{=}(1,1,0), \varepsilon_2{=}(0,2,1), \varepsilon_3{=}(1,0,-1)\big)$ defined at previous question. Then:

$$\varepsilon_1 \in F \Longrightarrow s(\varepsilon_1) = \varepsilon_1, \qquad \varepsilon_2 \in F \Longrightarrow s(\varepsilon_2) = \varepsilon_2 \qquad \text{and} \qquad \varepsilon_3 \in G \Longrightarrow s(\varepsilon_3) = -\varepsilon_3$$

We hence get: $A' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

(d) Let A be the matrix of s in the standard basis as input and output basis. Write the formula which enables one to compute A. **We don't ask you to do the computation.**

Consider the transition matrix from $\mathcal{B}$ to $\mathcal{B}'$: $P = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & -1 \end{pmatrix}$.

Then we know that $A' = P^{-1}AP$, which leads to: $A = PA'P^{-1}$.

Exercise 6: Probabilities (5 points)

Let $p \in]0,1[$. Consider a random variable X which is geometric-distributed with parameter p.

1. Write explicitly the distribution of X.

$$X(\Omega) = \mathbb{N}^* \qquad \text{and} \qquad \forall n \in \mathbb{N}^*,\ P(X{=}n) = p(1-p)^{n-1} = pq^{n-1} \text{ (setting } q = 1-p)$$

2. Let $(k,n) \in (\mathbb{N}^*)^2$.

(a) Show that $P(X{>}n) = q^n$ where $q = 1 - p$.

Hint: you can start by writing $P(X{>}n) = \sum_{k=n+1}^{+\infty} P(X{=}k)$ or, alternatively, $P(X{>}n) = 1 - \sum_{k=1}^{n} P(X{=}k)$.

$$P(X{>}n) = \sum_{k=n+1}^{+\infty} P(X{=}k) = \sum_{k=n+1}^{+\infty} pq^{k-1} = pq^n \sum_{m=0}^{+\infty} q^m \quad \text{by setting } m = k-1-n$$

Thus,

$$P(X{>}n) = pq^n \times \frac{1}{1-q} = q^n \qquad \text{since} \qquad 1 - q = p$$

(b) Explain why $P(X{=}n{+}k \ \cap\ X{>}n) = P(X{=}n{+}k)$.

If the property "$X{=}n{+}k$" is true, then the property "$X{>}n$" is also true. In other words, if we define the sets:

$$A = \{\omega \in \Omega, X(\omega) = n + k\} \qquad \text{and} \qquad B = \{\omega \in \Omega, X(\omega) > n\}$$

then: $A \subset B \Longrightarrow A \cap B = A \Longrightarrow P(A \cap B) = P(A)$.

(c) Compute the conditional probability $P(X{=}n{+}k \mid X{>}n)$. Compare your result with the value of $P(X{=}k)$.

According to the definition of a conditional probability, $P(X{=}n{+}k \mid X{>}n) = \dfrac{P(X{=}n{+}k \ \cap \ X{>}n)}{P(X{>}n)}$

Thus, using questions 2.a and 2.b.,

$$P(X{=}n{+}k \mid X{>}n) = \frac{P(X{=}n{+}k)}{q^n} = \frac{pq^{n+k-1}}{q^n} = pq^{k-1}$$

We can see that $P(X{=}n{+}k \mid X{>}n) = P(X{=}k)$.

(d) Explain why we say that the distribution of X is "memoryless".

Let us refer to the example of the hacker sending phishing emails to get Visa card numbers, where X is the number of messages he sends before getting a first answer.

If, after sending n messages, he still got no answers, we know that $X{>}n$. Then the probability of getting his first answer at the k^{th} next message is $P(X{=}n{+}k \mid X{>}n)$.

Yet, this probability is the same as $P(X{=}k)$. That is to say, the fact that he had no answers at his first n messages does not change the hacker's situation: it is the same as the initial situation.

3. Consider a random variable Y such that

$$Y(\Omega) = \mathbb{N}^* \qquad \text{and} \qquad \forall (k,n) \in (\mathbb{N}^*)^2 \,,\ P(Y{=}n{+}k \mid Y{>}n) = P(Y{=}k)$$

Let (p_n) be the sequence defined for all $n \in \mathbb{N}^*$ by: $\quad p_n = P(Y{=}n)$.

(a) Express $P(Y{>}1)$ as a function of p_1.

Since $Y(\Omega) = \mathbb{N}^*$, the complement of "$Y{>}1$" is "$Y{=}1$". Hence, $P(Y{>}1) = 1 - P(Y{=}1) = 1 - p_1$.

(b) By using the events "$Y{>}1$", "$Y{=}1$" and "$Y{=}2$", express $\dfrac{p_2}{p_1}$ as a function of p_1.

By writing the hypothesis in the case $k = n = 1$, we get:

$$\frac{P(Y{=}1{+}1 \cap Y{>}1)}{P(Y{>}1)} = P(Y{=}1) \Longrightarrow \frac{p_2}{1-p_1} = p_1 \Longrightarrow \frac{p_2}{p_1} = 1 - p_1$$

(c) Similarly, for all $n \in \mathbb{N}^*$, by using the events "$Y{>}1$", "$Y{=}n$" and "$Y{=}n+1$", find a simple expression of $\dfrac{p_{n+1}}{p_n}$.

Using the hypothesis, we get:

$$\frac{P(Y{=}1{+}n \cap Y{>}1)}{P(Y{>}1)} = P(Y{=}n) \Longrightarrow \frac{p_{n+1}}{1-p_1} = p_n \Longrightarrow \frac{p_{n+1}}{p_n} = 1 - p_1$$

(d) Deduce the value of p_n as a function of n. How do we call the distribution of Y?

The sequence (p_n) is a geometric sequence with common ratio $1 - p_1$. Thus, for all $n \in \mathbb{N}^*$,

$$p_n = p_1(1-p_1)^{n-1}$$

The random variable Y is hence geometric-distributed with parameter p_1.

The only "memoryless" distributions of random variables taking their values in $\mathbb{N}^*$ are the geometric distributions.