EPITA

Mathematics

Final exam S3

December 2021

Duration: 3 hours

Name:

First name:

Class:

MARK:

Instructions:

- Documents and pocket calculators are not allowed.
- Write your answers on the stapled sheets provided for answering. No other sheet will be corrected.
- Please, do not use lead pencils for answering.

Exercise 1 (3.5 points)

In $E = \mathbb{R}^3$ together with its standard basis $\mathcal{B}$, consider the family $\mathcal{F} = \{\varepsilon_1=(1,-1,2),\ \varepsilon_2=(2,1,-3),\ \varepsilon_3=(4,-1,1)\}$.

1. Is this family a basis of E? If not, extract a maximal independent sub-family and complete it to get a basis of E. The basis that you get will be denoted by $\mathcal{B}'$.

2. Find the coordinates in $\mathcal{B}'$ of the vector $u = (3, 3, -8)$.

3. Write the transition matrix from the standard basis $\mathcal{B}$ to the basis $\mathcal{B}'$.

Exercise 2 (3.5 points)

Let $f \in \mathcal{L}\left(\mathbb{R}^3\right)$ be defined by its matrix in the standard basis as input and output basis: $A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ -3 & 3 & -2 \end{pmatrix}$

1. Show that f is a projector.

2. Find a basis of $\mathrm{Ker}(f)$.

3. Find a basis of $\mathrm{Im}(f)$.

4. Write accurately the rank-nullity theorem and check that your results are consistent with this theorem.

5. Let $\mathcal{B}'$ be the concatenation of the bases of $\operatorname{Ker}(f)$ and of $\operatorname{Im}(f)$ that you have found in questions 2 and 3. We accept without proof that $\mathcal{B}'$ is a basis of $\mathbb{R}^3$.
   Write the matrix of f in $\mathcal{B}'$ as input and output basis.

Exercise 3 (5 points)

Consider the linear map $f : \left\{ \begin{array}{ccc} \mathbb{R}_2[X] & \longrightarrow & \mathbb{R}^3 \\ Q & \longmapsto & \big(Q(0),\, Q(1),\, Q(2)\big) \end{array} \right.$

1. Find the matrix of f in the standard bases $\{1, X, X^2\}$ as input and $\{(1,0,0),(0,1,0),(0,0,1)\}$ as output basis. Let us denote this matrix by A.

2. In $\mathbb{R}_2[X]$ consider the polynomials $Q_0 = \dfrac{(X-1)(X-2)}{2}$, $Q_1 = -X(X-2)$ and $Q_2 = \dfrac{X(X-1)}{2}$.
   (a) Write the values of $Q_i(0)$, $Q_i(1)$ and $Q_i(2)$ for each $i \in \{0,1,2\}$ (no need to show all the details of your computations).

   (b) Show that the family $\mathcal{B}' = \{Q_0, Q_1, Q_2\}$ is a basis of $\mathbb{R}_2[X]$.

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(c) Find the matrix of f in $\mathcal{B}'$ as input basis and the standard basis as output basis. This matrix is denoted by A'.

(d) Show that f is bijective and write the matrix of f^{-1} in the standard basis of $\mathbb{R}^3$ as input basis, $\mathcal{B}'$ as output basis.

3. Deduce the matrix of f^{-1} in the standard bases as input and output bases.

Exercise 4 (4.5 points)

Consider the two numerical sequences (x_n) and (y_n) defined by:

$$x_0 = 3, \quad y_0 = -2 \quad \text{and} \quad \forall n \in \mathbb{N}, \begin{cases} x_{n+1} &= \frac{3}{4}x_n - \frac{1}{8}y_n \\ y_{n+1} &= -\frac{1}{2}x_n + \frac{3}{4}y_n \end{cases}$$

In the vector space $\mathbb{R}^2$ together with its standard basis $\mathcal{B} = \{(1,0), (0,1)\}$, let us define the sequence (u_n) by: $u_n = (x_n, y_n)$. For example, $u_0 = (3, -2)$.

1. Find $f \in \mathcal{L}(\mathbb{R}^2)$ such that for every $n \in \mathbb{N}$, $u_{n+1} = f(u_n)$.

2. Consider the following basis of $\mathbb{R}^2$: $\mathcal{B}' = \{\varepsilon_1{=}(1,-2),\ \varepsilon_2{=}(1,2)\}$. Write the transition matrix from $\mathcal{B}$ to $\mathcal{B}'$. Then find the coordinates of u_0 in $\mathcal{B}'$.

3. Write the matrix of f in basis $\mathcal{B}'$ as input and output basis.

4. For every $n \in \mathbb{N}$, let $X'_n = \begin{pmatrix} x'_n \\ y'_n \end{pmatrix}$ be the column matrix containing the coordinates of u_n in basis $\mathcal{B}'$. Express X'_{n+1} as a function of X'_n.

5. Deduce the coordinates of u_n in the basis $\mathcal{B}'$, then $\lim\limits_{n \to +\infty} x_n$ and $\lim\limits_{n \to +\infty} y_n$.

Exercise 5 (4 points)

Let $A = \begin{pmatrix} -1 & -1 & -1 \\ 1 & 2 & 0 \\ 5 & 2 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & -2 & 2 \\ 10 & -9 & 10 \\ 6 & -6 & 7 \end{pmatrix}$

1. Find the characteristic polynomials of A and B, **in a factorized form**.
   Check that the eigenvalues of A are 1 and 2, and that those of B are -1 and 1.

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2. Are the matrices A and B diagonalizable in $\mathscr{M}_3(\mathbb{R})$? If they are, find the matrices P and D.
N.B.: the dimension of each required eigen subspace must be determined by exhibiting a basis. The latter basis must be deduced from a clear reasoning, and not by randomly picking particular values.

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