EPITA

Mathematics

Final exam (S3)

December 2018

Name :

First name :

Class :

MARK :

Exercise 1 (5 points)

Let $A = \begin{pmatrix} -1 & 2 & 1 \\ 2 & -1 & -1 \\ -4 & 4 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 & -1 \\ -3 & 4 & -3 \\ -1 & 1 & 0 \end{pmatrix}$.

Are A and B diagonalizable in $\mathscr{M}_3(\mathbb{R})$? If they are, determine D and P.

N.B. : the bases of the eigenspaces must be deduced from a clear reasoning, and not by randomly picking particular values.

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Exercise 2 (3 points)

Let $a \in \mathbb{R}$ and $A = \begin{pmatrix} 3-a & -5+a & a \\ -a & a-2 & a \\ 5 & -5 & -2 \end{pmatrix}$.

Study the diagonalizability of A in $\mathscr{M}_3(\mathbb{R})$ depending on the value of a.

N.B. : When A is diagonalizable, the eigenbasis is not required.

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Exercise 3 (4 points)

1. Let $f : \left\{ \begin{array}{ccc} \mathbb{R}^3 & \longrightarrow & \mathbb{R}^2 \\ (u, v, w) & \longmapsto & (2u - w; 3u + v + 2w) \end{array} \right.$.

   Determine the matrix of f in the standard bases of the input and output spaces.

2. Let $E = \mathbb{R}_2[X]$ and $f : \left\{ \begin{array}{ccc} E & \longrightarrow & E \\ P(X) & \longmapsto & 2P(X+1) + P(X-1) - 2P(X) \end{array} \right.$.

   Determine the matrix of f in the standard basis $(1, X, X^2)$ of $\mathbb{R}_2[X]$.

Exercise 4 (4 points)

Let E and F be two vector spaces over $\mathbb{R}$, $f \in \mathscr{L}(E,F)$ and let $X = (x_1, \ldots, x_n)$ be a family of vectors of E. Show that :

1. $f(\text{Span}(X)) = \text{Span}\big(f(X)\big)$.

2. $[f \text{ surjective and } \text{Span}(X) = E] \implies \text{Span}\big(f(X)\big) = F$.

3. $[f \text{ injective and } X \text{ linearly independent}] \implies f(X)$ linearly independent.

Exercise 5 (3 points)

1. Prove that any polynomial of odd degree with real coefficients has at least one real root.

2. Let $A \in \mathscr{M}_n(\mathbb{R})$ such that $A^2 + A + I = 0$ $(*)$.

   a. Let $\lambda \in \mathrm{Sp}_{\mathbb{R}}(A)$. Show that $\lambda^2 + \lambda + 1 = 0$.

   b. Show that a matrix $A \in \mathscr{M}_3(\mathbb{R})$ cannot satisfy the equation $(*)$.

Exercise 6 (2 points)

Let $n \in \mathbb{N}^*$, $(a_1, \ldots, a_n) \in \mathbb{R}^n$, $(b_1, \ldots, b_n) \in \mathbb{R}^n$. Compute the determinant of size $n+1$: $\Delta_n = \begin{vmatrix} 1 & 1 & 1 & \cdots & \cdots & 1 \\ b_1 & a_1 & a_1 & \cdots & \cdots & a_1 \\ b_1 & b_2 & a_2 & \cdots & \cdots & a_2 \\ \vdots & \vdots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \vdots \\ b_1 & b_2 & \cdots & \cdots & b_n & a_n \end{vmatrix}$.