EPITA

Mathematics

Final exam (S3)

December 2017

Name :

First name :

Class :

MARK :

Exercise 1 (6 points)

Let $A = \begin{pmatrix} 3 & -3 & 2 \\ -1 & 5 & -2 \\ -1 & 3 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 2 & -1 \\ 3 & -2 & 0 \\ -2 & 2 & 1 \end{pmatrix}$.

Are A and B diagonalizable in $\mathscr{M}_3(\mathbb{R})$? If they are, determine D and P.

N.B. : the bases of the eigenspaces must be deduced from a clear reasoning, and not by randomly picking particular values.

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Exercise 2 (4 points)

Let $a \in \mathbb{R}$ and $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ a^2 - a & -a-1 & a^2+1 \end{pmatrix}$.

Study the diagonalizability of A in $\mathscr{M}_3(\mathbb{R})$ depending on the value of a.

N.B. : when A is diagonalizable, the eigenbasis is not required.

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Exercise 3 (4 points)

1. Let $n \in \mathbb{N}$ and $f : \left\{ \begin{array}{ccc} \mathbb{R}_n[X] & \longrightarrow & \mathbb{R} \\ P & \longmapsto & \displaystyle\int_0^1 P(t)\mathrm{d}t \end{array} \right.$.

   Determine the matrix of f in the standard bases of the input and output spaces.

2. Let $E = \mathbb{R}_3[X]$ and $f : \left\{ \begin{array}{ccc} E & \longrightarrow & E \\ P(X) & \longmapsto & (X^2-1)P''(X) + 2XP'(X) \end{array} \right.$.

   Determine the matrix of f in the standard basis $(1, X, X^2, X^3)$ of $\mathbb{R}_3[X]$.

Exercise 4 (4 points)

Let E be a finite dimensional vector space over $\mathbb{R}$, F and G be two supplementary linear subspaces of E. Let $\mathscr{B} = (e_1, \ldots, e_p)$ be a basis of F and $\mathscr{B}' = (f_1, \ldots, f_q)$ be a basis of G. Prove, WITHOUT referring to the property $\dim(E) = \dim(F) + \dim(G)$, that the concatenation of $\mathscr{B}$ and $\mathscr{B}'$ is a basis of E.

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Exercise 5 (3 points)

Calculate the determinant of the following matrix of order $n \geqslant 2$: $\begin{pmatrix} 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 1 & & \ddots & 1 & 0 \\ 0 & 1 & \dots & 1 & 1 \end{pmatrix}$.