

# Final exam n°1

Duration : three hours

Documents and calculators not allowed

Name :

First Name :

Class :

## Instructions :

- no sheets other than the stapled ones provided for answers shall be corrected.
- answers written using lead pencils shall not be corrected.

## Exercise 1 (4 points)

1. Using the ratio test (D'Alembert's test), determine the nature of the series  $\sum u_n$  where, for all  $n \in \mathbb{N}^*$ ,  $u_n = \frac{10^n}{n 4^{2n+1}}$ .

2. Using the root test (Cauchy's test), determine the nature of the series  $\sum v_n$  where, for all  $n \geq 2$ ,  $v_n = \frac{n}{(\ln(n))^n}$ .

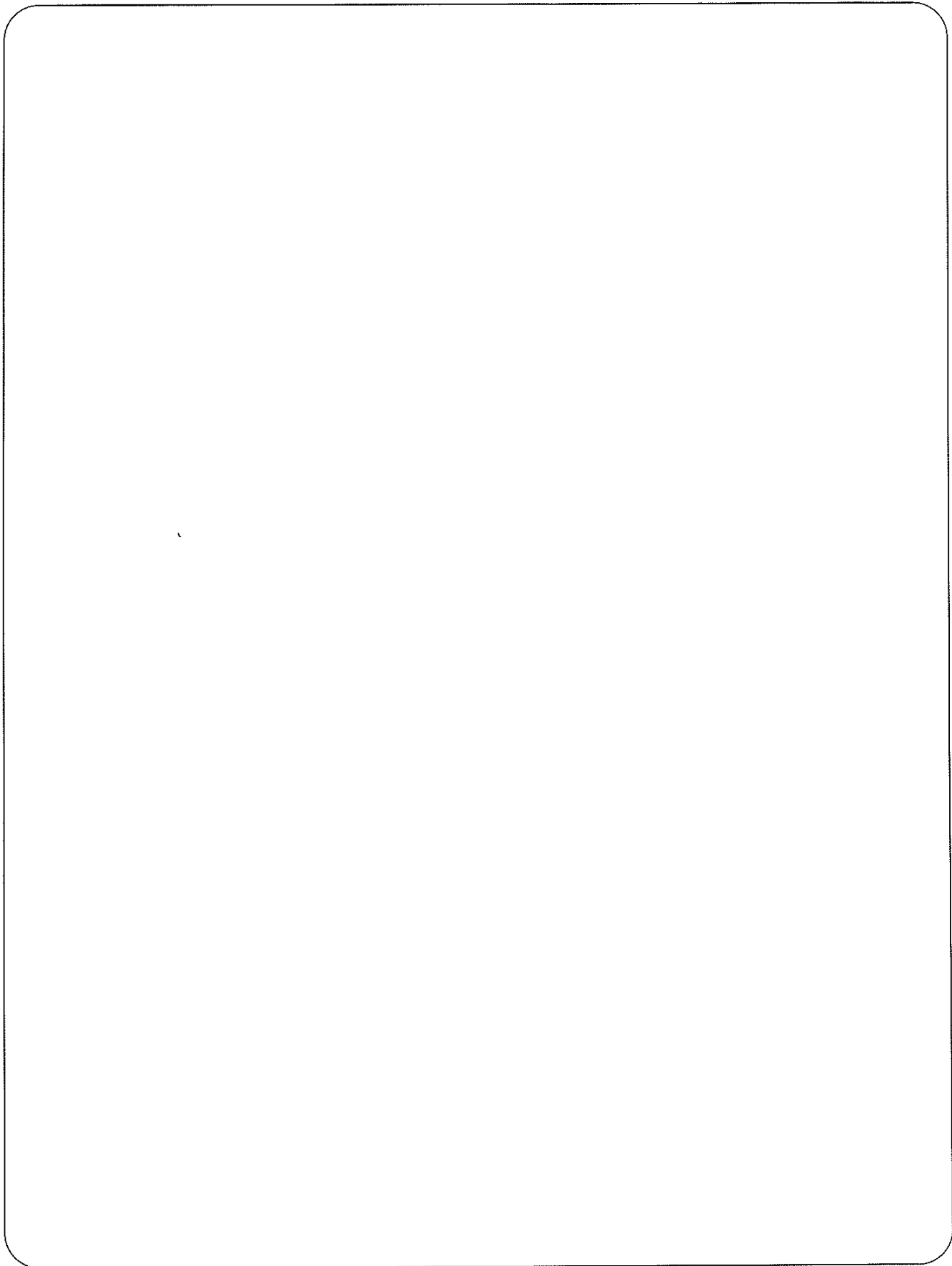
### Exercise 2 (4 points)

Let  $A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 1 \end{pmatrix}$  and  $B = \begin{pmatrix} 3 & -1 & 1 \\ 7 & -5 & 1 \\ 6 & -6 & 2 \end{pmatrix}$ .

Are  $A$  and  $B$  diagonalizable in  $\mathcal{M}_3(\mathbb{R})$ ? If so, determine  $D$  et  $P$ .

N.B. : The bases of the eigenspaces must be deduced from clear reasoning, and not by randomly picking particular values.

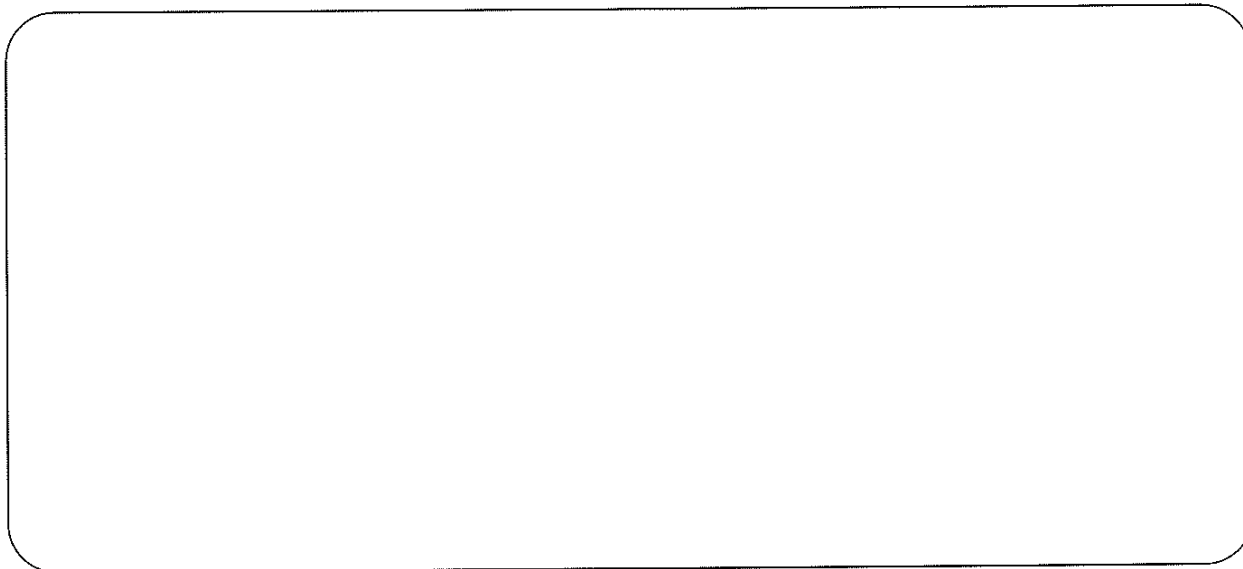
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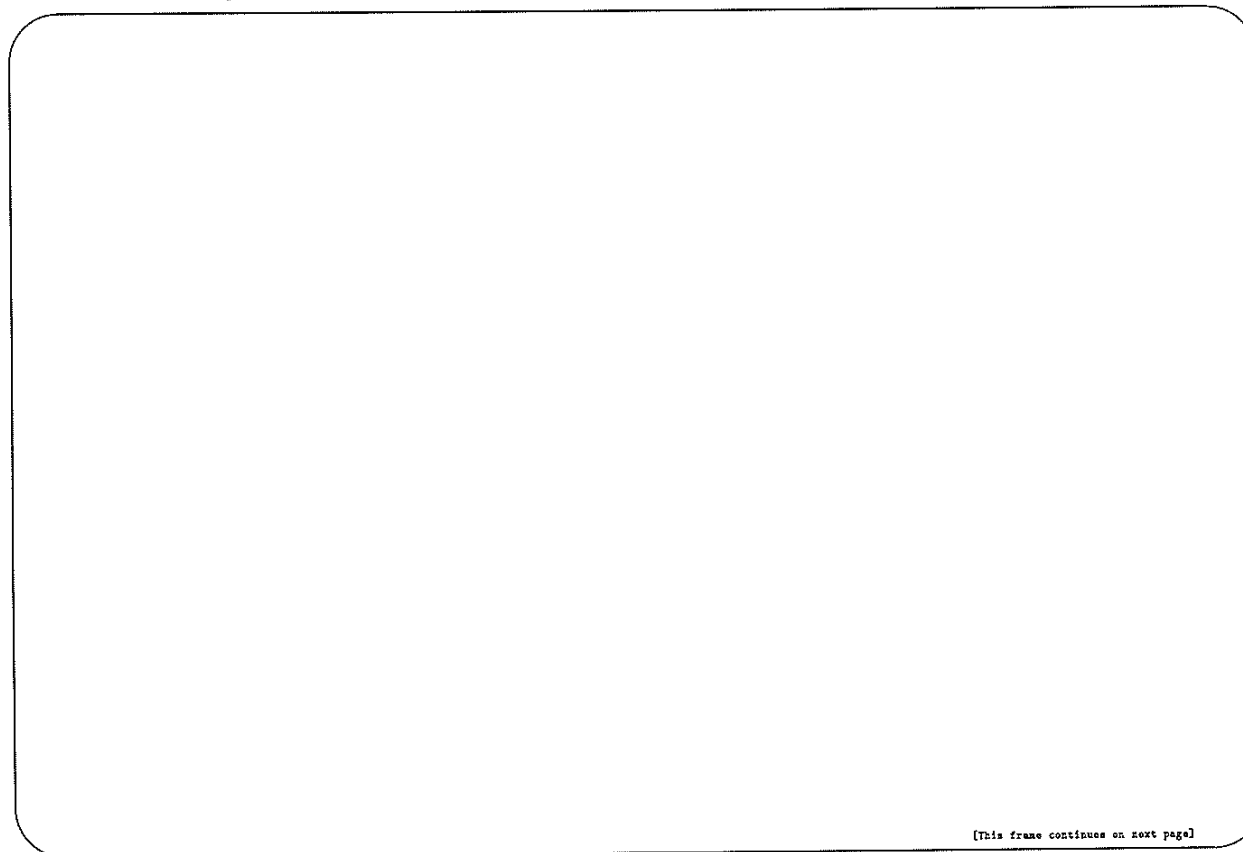
### Exercise 3 (3.5 points)

Let  $a \in \mathbb{R}$  and  $A = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 2 & 1 \\ 2-a & a-2 & a \end{pmatrix}$ .

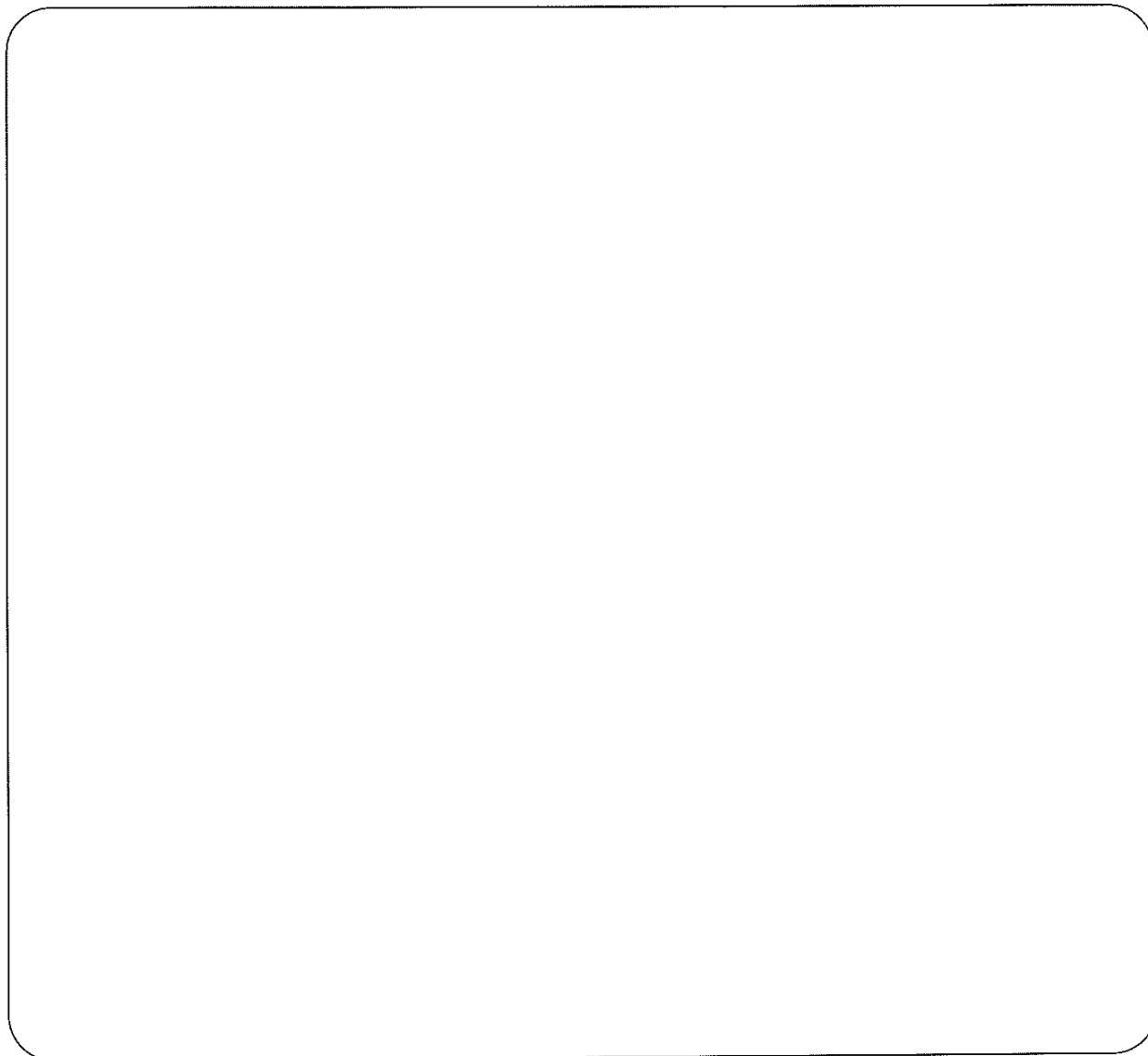
1. Determine the characteristic polynomial of  $A$ , denoted  $P_A$ , by choosing as first transformation :  
 $C_1 \leftarrow C_1 + C_2$ .



2. Study the diagonalizability of  $A$  in  $\mathcal{M}_3(\mathbb{R})$  depending on the value of  $a$ .  
N.B. : when  $A$  is diagonalizable, the diagonalization is not required.



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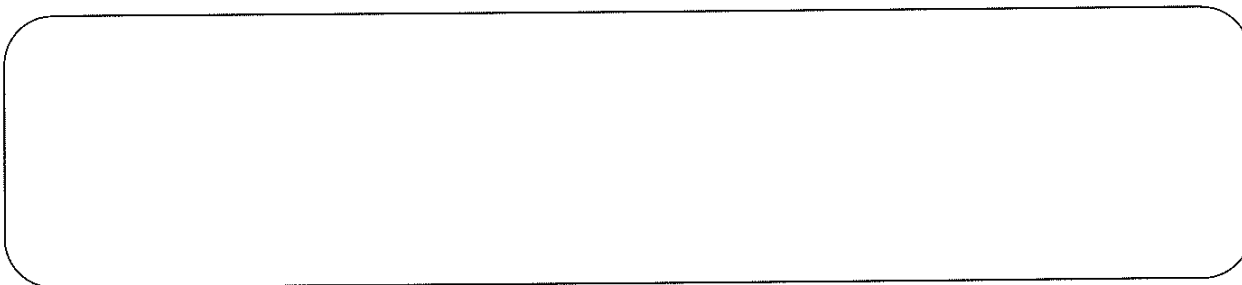


**Exercise 4 (3.5 points)**

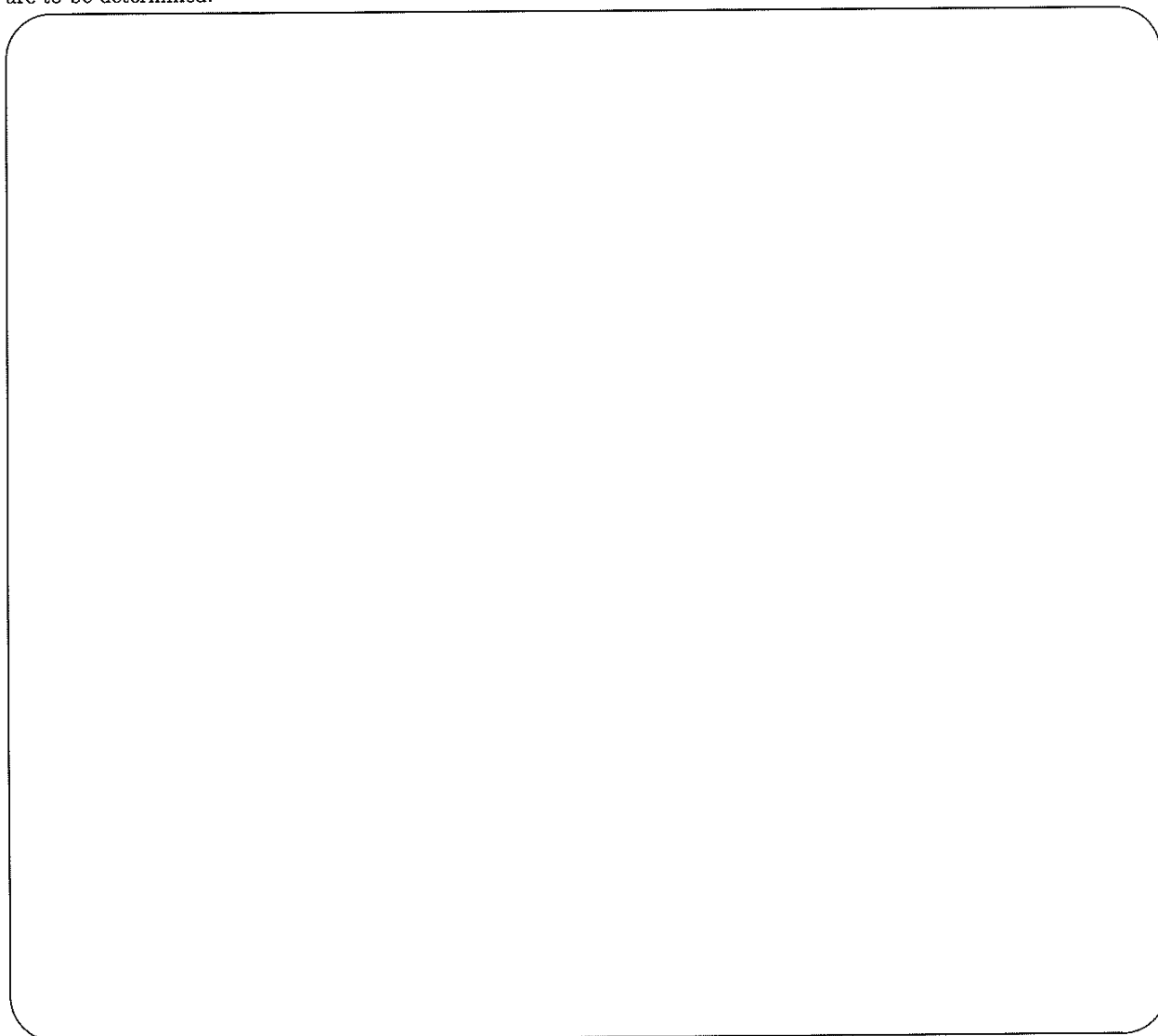
We want to study the following linear differential system : 
$$\begin{cases} x'(t) = x(t) + 8y(t) \\ y'(t) = x(t) + 3y(t) \end{cases} .$$

Let's denote  $X(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ .

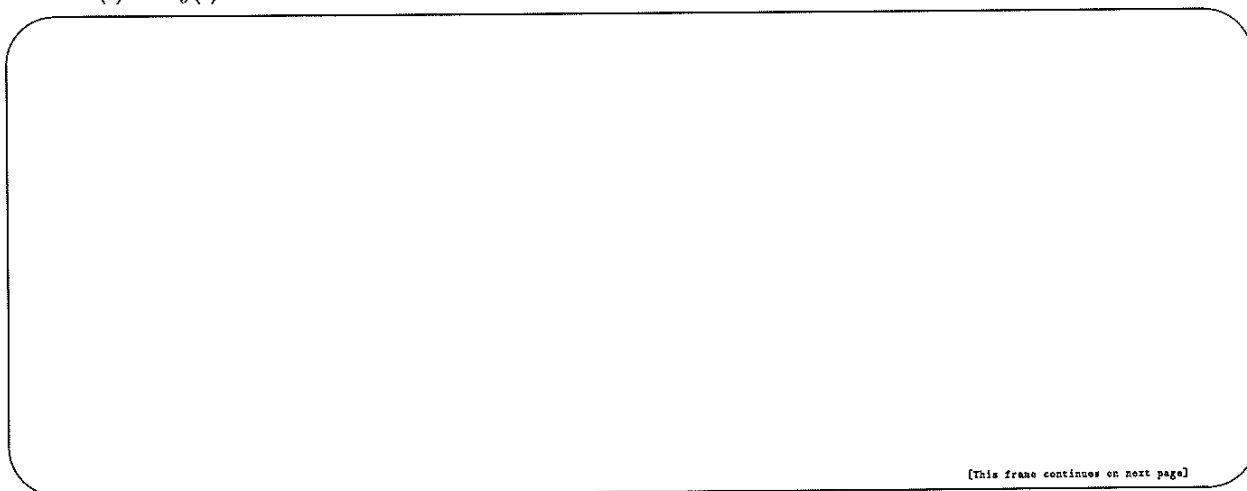
1. Determine  $A \in \mathcal{M}_2(\mathbb{R})$  such that  $X'(t) = AX(t)$ .



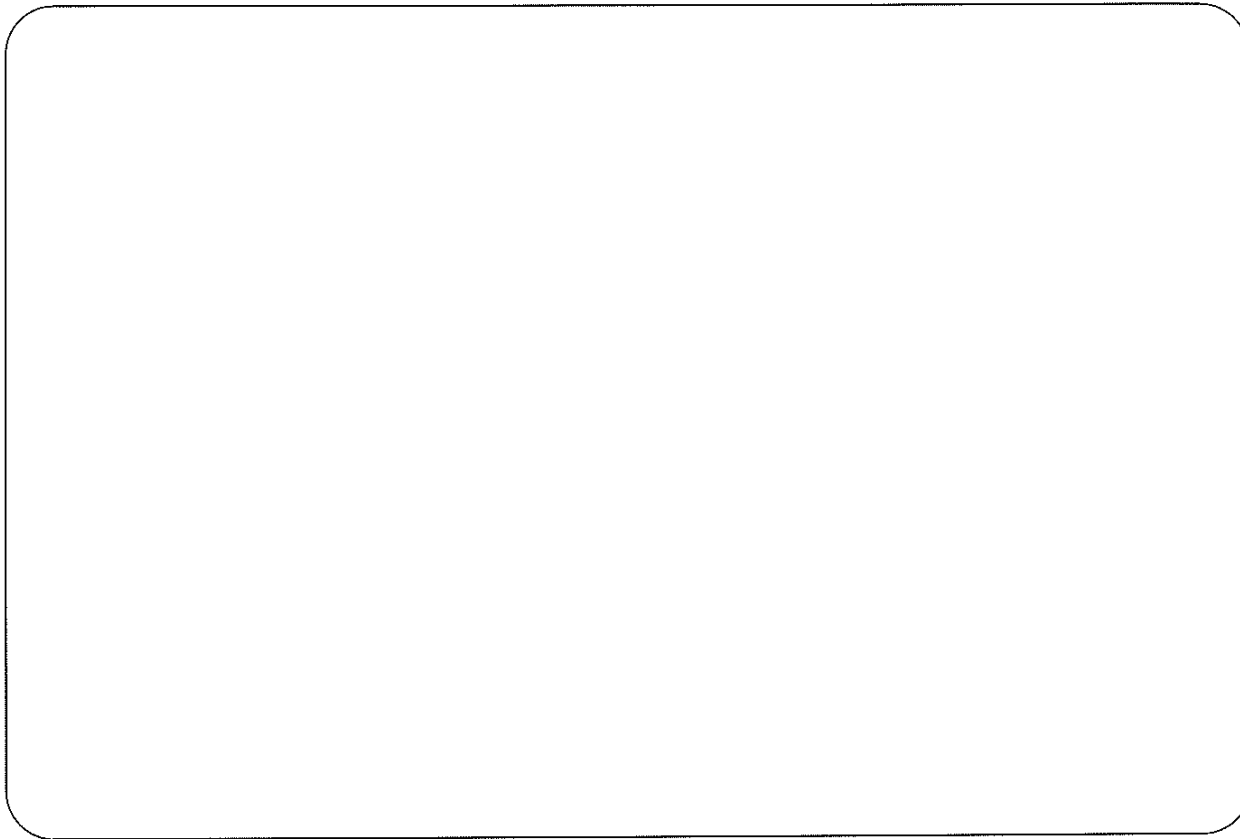
2. Diagonalize  $A$ , by exhibiting  $D$  and  $P$ . The matrix  $D$  will be of the form  $D = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ , with  $a < b$ , where  $a$  and  $b$  are to be determined.



3. [Please check that you have chosen  $a < b$  in the matrix  $D$  obtained in the previous question]. From the previous questions, deduce  $x(t)$  and  $y(t)$  as functions of  $t$ .



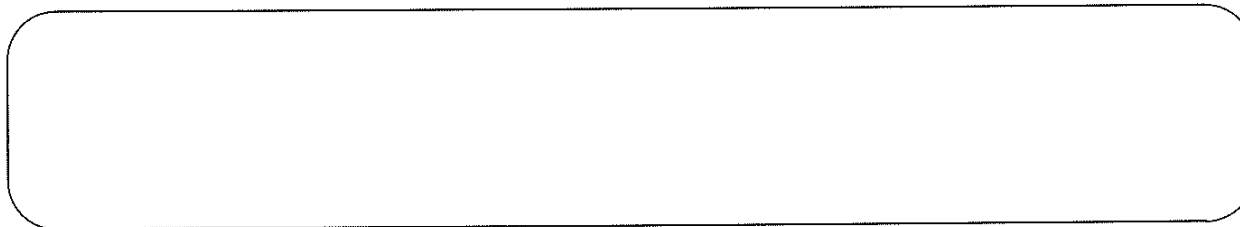
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**Exercise 5 (4 points)**

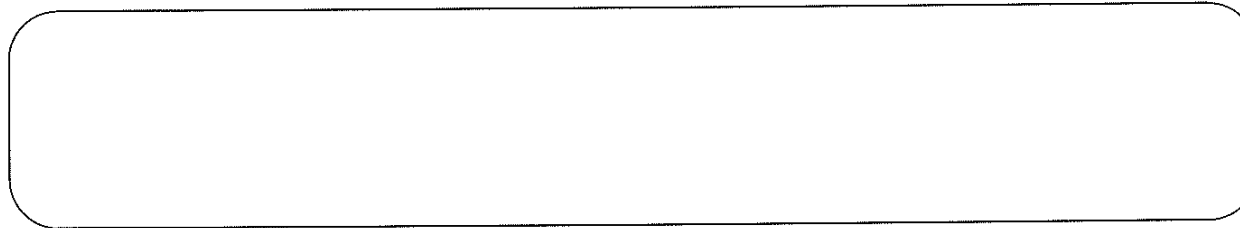
1. Let  $A = \begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix}$  and  $f : \begin{cases} \mathcal{M}_2(\mathbb{R}) & \rightarrow \mathcal{M}_2(\mathbb{R}) \\ M & \mapsto AM \end{cases}$ . Determine the matrix of  $f$  with respect to the standard basis

$\mathcal{B} = \left( E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right)$  of  $\mathcal{M}_2(\mathbb{R})$ .



2. Let  $\Delta : \begin{cases} \mathcal{M}_2(\mathbb{R}) & \rightarrow \mathbb{R}_2[X] \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \mapsto (a+d)X^2 + (b+c)X + d - c \end{cases}$

Determine the matrix of  $\Delta$  with respect to the standard bases of  $\mathcal{M}_2(\mathbb{R})$  and  $\mathbb{R}_2[X]$ .



### Exercise 6 (2 points)

Let  $(a_1, \dots, a_n) \in \mathbb{R}^n$ . Express the following determinant (over factorized form), indicating the transformations :

$$\begin{vmatrix} a_1 & a_1 & a_1 & \dots & \dots & a_1 \\ a_1 & a_2 & a_2 & \dots & \dots & a_2 \\ a_1 & a_2 & a_3 & \dots & \dots & a_3 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ a_1 & a_2 & a_3 & \dots & \dots & a_n \end{vmatrix}$$

