

# Correction of the midterm exam S2 2023

## Exercise 1: polynomials

Consider the polynomial  $P(X) = X^6 - X^5 - 3X^4 + 7X^3 + 14X^2 + 6X$ .

1. Show that  $-1$  is a root of  $P$  and find its (exact) order of multiplicity.

- $P(-1) = 1 + 1 - 3 - 7 + 14 - 6 = 0$ .  $-1$  is a root of  $P$ .
- $P'(X) = 6X^5 - 5X^4 - 12X^3 + 21X^2 + 28X + 6$ . Hence  $P'(-1) = -6 - 5 + 12 + 21 - 28 + 6 = 0$
- $P''(X) = 30X^4 - 20X^3 - 36X^2 + 42X + 28$ . Hence,  $P''(-1) = 30 + 20 - 36 - 42 + 28 = 0$ .
- $P'''(X) = 120X^3 - 60X^2 - 72X + 42$ . Hence,  $P'''(-1) = -66 \neq 0$ .

Finally,  $-1$  is a root of  $P$  of multiplicity 3.

2. What can we deduce in terms of divisibility?

We can deduce that  $(X + 1)^3 \mid P$  and that  $(X + 1)^4 \nmid P$ .

3. Using only one euclidean division, factorize  $P$  as a product of irreducible polynomials in  $\mathbb{R}[X]$ .

Remind that the irreducible polynomials in  $\mathbb{R}[X]$  are only those of degree 1 and of degree 2 with a negative discriminant.

Let us do the euclidean division of  $P$  by  $(X + 1)^3 = X^3 + 3X^2 + 3X + 1$ . We find a zero reminder and the quotient  $Q = X^3 - 4X^2 + 6X$ .

Thus,  $P = (X + 1)^3 (X^3 - 4X^2 + 6X) = (X + 1)^3 X (X^2 - 4X + 6)$ . The polynomial  $X^2 - 4X + 6$  has a discriminant equal to  $-8 < 0$ . It is hence irreducible in  $\mathbb{R}[X]$ . Finally,  $P$  can be written as this product of irreducible polynomials in  $\mathbb{R}[X]$ :  $(X + 1)^3 X (X^2 - 4X + 6) = (X + 1)(X + 1)(X + 1)X(X^2 - 4X + 6)$ .

## Exercise 2: differential equations

The questions of the exercise are independent.

1. Consider the differential equation  $(E_1) : (x + 1)y' - 2y = (x + 1)^3 \cos(3x)$  on  $I = ]-1, +\infty[$ .

(a) Solve  $(E_1)$  on  $I$ .

- The solutions of the homogeneous equation are the functions of the form

$$y_0(x) = ke^{-\int \frac{-2}{x+1} dx} = ke^{2 \ln(x+1)} = k(x + 1)^2 \text{ with } k \in \mathbb{R}$$

- Let us find a particular solution of  $(E_1)$  as a function of the form  $y_p(x) = k(x)(x + 1)^2$  (variation of the parameter).

Then  $y'_p(x) = k'(x)(x + 1)^2 + 2k(x)(x + 1)$ . This leads to:

$$\begin{aligned} y_p(x) \text{ solution of } (E_1) &\iff (x + 1)y'_p(x) - 2y_p(x) = (x + 1)^3 \cos(3x) \\ &\iff (x + 1)^3 k'(x) + 2k(x)(x + 1)^2 - 2k(x)(x + 1)^2 = (x + 1)^3 \cos(3x) \end{aligned}$$

Hence,  $k'(x) = \cos(3x)$ . Let us choose, for example  $k(x) = \frac{1}{3} \sin(3x)$ .  $y_p(x) = \frac{(x + 1)^2}{3} \sin(3x)$  is hence a particular solution of  $(E_1)$ .

$$\text{Finally, } S = \left\{ \begin{array}{ll} I & \longrightarrow \mathbb{R} \\ x & \longmapsto k(x + 1)^2 + \frac{(x + 1)^2}{3} \sin(3x) \end{array} ; k \in \mathbb{R} \right\}$$

(b) Find the solutions of  $(E_1)$  satisfying to:  $y(0) = 1$ .

We have to choose the parameter  $k$  such that  $y(0) = 1$ . Then,  $y(0) = k(0 + 1)^2 + \frac{(0+1)^2}{3} \sin(3 \times 0) = k$ . Thus,  $k = 1$  and there exists only one function  $y$ , solution de  $(E_1)$  such that  $y(0) = 1$ . It is the function  $y : x \mapsto (x + 1)^2 + \frac{(x + 1)^2}{3} \sin(3x)$ .

2. Consider the differential equation  $(E_2) : y'' + 4y' + 13y = (25x^2 + 16x + 2)e^{2x}$  on  $J = \mathbb{R}$ .

(a) Show that  $y_p : x \mapsto x^2 e^{2x}$  is a particular solution of  $(E_2)$ .

We compute  $y_p'(x) = (2x + 2x^2) e^{2x}$  and  $y_p''(x) = (2 + 4x + 4x + 4x^2) e^{2x}$ . By injecting these expressions in  $(E_2)$ , we get:

$$y_p''(x) + 4y_p'(x) + 13y_p(x) = (2 + 8x + 4x^2 + 8x + 8x^2 + 13x^2)e^{2x} = (25x^2 + 16x + 2)e^{2x}$$

$y_p$  is hence a solution of  $(E_2)$ .

(b) Find the solution set of  $(E_2)$ .

- The characteristic equation associated to  $(E_2)$  is  $(C) r^2 + 4r + 13 = 0$ . Its discriminant is  $\Delta = -36$ . The roots of  $(C)$  are hence  $r_1 = \frac{-4 + 6i}{2} = -2 + 3i$  and  $r_2 = -2 - 3i$ .

- We can hence deduce that  $S = \left\{ \begin{array}{l} \mathbb{R} \rightarrow \mathbb{R} \\ x \mapsto e^{-2x} (k_1 \cos(3x) + k_2 \sin(3x)) + (25x^2 + 16x + 2)e^{2x} \end{array} ; (k_1, k_2) \in \mathbb{R}^2 \right\}$

### Exercise 3: local analysis of functions

1. Let  $f$  and  $g$  be two real functions defined on  $\mathbb{R}$ . Let  $a \in \mathbb{R} \cup \{+\infty, -\infty\}$ . Write the mathematical definitions of:  $f(x) \sim g(x)$  and  $f(x) = o(g(x))$  as  $x$  approaches  $a$ .

- $f(x) \sim g(x) \iff f(x) = g(x)(1 + \varepsilon(x))$  with  $\lim_{x \rightarrow a} \varepsilon(x) = 0$ . We can also write:  $f(x) \sim g(x) \iff \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = 1$ .

- $f(x) = o(g(x)) \iff f(x) = g(x)\varepsilon(x)$  with  $\lim_{x \rightarrow a} \varepsilon(x) = 0$ . We can also write:  $f(x) = o(g(x)) \iff \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = 0$ .

2. Find simple equivalents (others than the function  $f$  itself) of  $f(x) = 3x^3 - 2x^2 + 6x$  in  $a = 0$  AND in  $a = +\infty$ . Justify your answers.

- Note that  $\frac{3x^3 - 2x^2 + 6x}{6x} = \frac{x^2}{2} - \frac{x}{3} + 1$ . Hence,  $\lim_{x \rightarrow 0} \frac{f(x)}{6x} = 1$ . Finally,  $f(x) \sim 6x$  as  $x$  approaches 0.

- Note that  $\frac{3x^3 - 2x^2 + 6x}{3x^3} = 1 - \frac{2}{3x^2} + \frac{2}{x^2}$ . Hence,  $\lim_{x \rightarrow +\infty} \frac{f(x)}{3x^3} = 1$ . Finally,  $f(x) \sim 3x^3$  as  $x$  approaches  $+\infty$ .

3. Let  $h$  and  $k$  be two functions such that, as  $x$  approaches 0,

$$h(x) = 1 + 2x + x^2 - 3x^3 + o(x^3) \quad \text{and} \quad k(x) = -x + 3x^2 + o(x^2)$$

(a) Find equivalents in 0, the simplest as possible, of:  $h(x)$  (don't justify),  $k(x)$  (don't justify) and  $xh(x) + k(x)$  (justify).

We have  $h(x) \sim 1$  in 0 and  $k(x) \sim -x$  in 0. Furthermore,  $xh(x) + k(x) = 5x^2 + o(x^2)$ . Hence,  $xh(x) + k(x) \sim 5x^2$  in 0.

(b) Do you have enough information to write the Taylor expansion of  $h(x) + k(x)$  at the order 1? And at the order 2? And at the order 3? Write explicitly the Taylor expansion when your answer is positive.

We can write TEs at the orders 1 and 2, but we don't have enough information for a TE at the order 3 (because of  $k$ ). The possible TEs are:

$$h(x) + k(x) = 1 + x + o(x) \quad (\text{order 1}) \quad \text{and} \quad h(x) + k(x) = 1 + x + 4x^2 + o(x^2) \quad (\text{order 2})$$

### Exercise 4: Taylor expansions

At each question, write all the basic Taylor expansions that you will use before answering the question.

1. Find the Taylor expansion in 0, at the order 3, of  $f(x) = \cos(x)e^{-2x}$ .

$$f(x) = \left(1 - \frac{x^2}{2} + o(x^3)\right) \times \left(1 - 2x + \frac{(-2x)^2}{2} + \frac{(-2x)^3}{3!} + o(x^3)\right)$$

Thus,

$$f(x) = \left(1 - \frac{x^2}{2} + o(x^3)\right) \times \left(1 - 2x + 2x^2 - \frac{4x^3}{3} + o(x^3)\right) = 1 - 2x + 2x^2 - \frac{4x^3}{3} - \frac{x^2}{2} + x^3 + o(x^3)$$

Finally,  $f(x) = 1 - 2x + \frac{3x^2}{2} - \frac{x^3}{3} + o(x^3)$ .

2. Find the Taylor expansion in 0, at the order 2, of  $g(x) = \sqrt{1+x}$ , using one of the five basic Taylor expansions.

$$g(x) = 1 + \frac{1}{2}x + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2}x^2 + o(x^2) = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + o(x^2)$$

3. Find the Taylor expansion in 0, at the order 2, of  $k(x) = \ln(1 + \sqrt{1+x})$ .

Using previous question:

$$h(x) = \ln\left(2 + \frac{x}{2} - \frac{x^2}{8} + o(x^2)\right) = \ln\left(2\left(1 + \frac{x}{4} - \frac{x^2}{16} + o(x^2)\right)\right) = \ln(2) + \ln\left(1 + \frac{x}{4} - \frac{x^2}{16} + o(x^2)\right)$$

Let  $u(x) = \frac{x}{4} - \frac{x^2}{16} + o(x^2)$  (which tends to 0 as  $x$  approaches 0). Then  $u^2(x) = \frac{x^2}{16} + o(x^2)$ . Since, as  $u$  approaches 0,  $\ln(1+u) = u - \frac{u^2}{2} + o(u^2)$ , we get:

$$h(x) = \ln(2) + \frac{x}{4} - \frac{x^2}{16} - \frac{x^2}{32} + o(x^2) = \ln(2) + \frac{x}{4} - \frac{3x^2}{32} + o(x^2)$$

### Exercise 5: finding limits

1. Find  $\lim_{x \rightarrow 0} \frac{e^x + e^{-x} - 2}{\sin(\frac{x}{2})}$ .

• Let  $N(x) = e^x + e^{-x} - 2$ .

Then,  $N(x) = \left(1 + x + \frac{x^2}{2} + o(x^2)\right) + \left(1 - x + \frac{(-x)^2}{2} + o(x^2)\right) - 2 = x^2 + o(x^2)$ . Thus,  $N(x) \sim x^2$  in 0.

• Let  $D(x) = \sin\left(\frac{x}{2}\right)$ . Using TEs, we get  $D(x) \sim \frac{x}{2}$ .

• We deduce that  $\frac{N(x)}{D(x)} \sim \frac{x^2}{\frac{x}{2}} = 2x$ . Since  $\lim_{x \rightarrow 0} 2x = 0$ , we get:  $\lim_{x \rightarrow 0} \frac{N(x)}{D(x)} = 0$

2. Find  $\lim_{x \rightarrow +\infty} \left(1 + \ln\left(1 + \frac{1}{x}\right)\right)^x$ .

$$\left(1 + \ln\left(1 + \frac{1}{x}\right)\right)^x = e^{x \ln\left(1 + \ln\left(1 + \frac{1}{x}\right)\right)} = e^{x \left(\ln\left(1 + \frac{1}{x} + o\left(\frac{1}{x}\right)\right)\right)} = e^{x \left(\frac{1}{x} + o\left(\frac{1}{x}\right)\right)} = e^{1 + o(1)}$$

Hence,  $\lim_{x \rightarrow +\infty} \left(1 + \ln\left(1 + \frac{1}{x}\right)\right)^x = e^1$

### Exercise 6: vector spaces 1

1. For each of these sets, say whether it is a vector space over  $\mathbb{R}$  or not. Justify your answers rigorously.

(a)  $E = \{(x, y) \in \mathbb{R}^2, x \leq y\}$

$E \subset \mathbb{R}^2$ , but  $E$  is not a linear subspace of  $\mathbb{R}^2$ .

For example, consider the vector  $u = (1, 2)$ . Then  $u \in E$  but  $-u = (-1, -2) \notin E$ . Thus,  $E$  is not closed for the scalar multiplication.

$E$  is hence not a vector space over  $\mathbb{R}$ .

(b)  $F = \{(x, y, z) \in \mathbb{R}^3, x - y = 0\}$ .

By definition,  $F \subset \mathbb{R}^3$ . Furthermore,  $0_{\mathbb{R}^3} = (0, 0, 0) \in F$  because  $0 - 0 = 0$ .

Let  $(u_1 = (x, y, z), u_2 = (x', y', z')) \in F^2$  and  $\lambda \in \mathbb{R}$ .

Then  $\lambda u_1 + u_2 = (\lambda x + x', \lambda y + y', \lambda z + z')$ . Note that  $(\lambda x + x') - (\lambda y + y') = \lambda(x - y) + (x' - y') = \lambda \times 0 + 0 = 0$  because  $u_1$  and  $u_2$  are elements of  $F$ .

This proves that  $\lambda u_1 + u_2 \in F$ .

Finally,  $F$  is a linear subspace of  $\mathbb{R}^3$ . It is hence a  $\mathbb{R}$ -vector space.

(c)  $G = \{P \in \mathbb{R}[X], X \mid P\}$

By definition,  $G \subset \mathbb{R}[X]$ . Furthermore,  $0_{\mathbb{R}[X]} = X \times 0_{\mathbb{R}}$ , that is,  $X \mid 0_{\mathbb{R}[X]}$ . Hence,  $0_{\mathbb{R}[X]} \in G$ .

Let  $(P_1, P_2) \in G^2$  and  $\lambda \in \mathbb{R}$ . Since  $X \mid P_1$  and  $X \mid P_2$ ,  $\exists (Q_1, Q_2) \in (\mathbb{R}[X])^2$  such that  $P_1 = XQ_1$  et  $P_2 = XQ_2$ . Thus,  $\lambda P_1 + P_2 = X(\lambda Q_1 + Q_2)$  which show that  $X \mid (\lambda P_1 + P_2)$ . Hence,  $\lambda P_1 + P_2 \in G$ .

Finally,  $G$  is a linear subspace of  $\mathbb{R}[X]$ . It is hence a  $\mathbb{R}$ -vector space.

2. In this question, you don't have to justify your answers.

Find a linear subspace of  $E$  (other than  $E$  and  $\{0_E\}$ ) in the following cases:

(a)  $E = \mathbb{R}^4$

For example:  $F = \{(x, y, z, t) \in \mathbb{R}^4, x + z + t = 0\}$ .

(b)  $E = \mathbb{R}^{\mathbb{R}}$

For example:  $F = \{f \in \mathbb{R}^{\mathbb{R}}, f(0) = 0\}$

(c)  $E = \{(u_n) \in \mathbb{R}^{\mathbb{N}}, (u_n) \text{ converges}\}$

For example:  $F = \{(u_n) \in \mathbb{R}^{\mathbb{N}}, (u_n) \text{ converges to } 0\}$

## Exercise 7: vector spaces 2

The questions are independent.

1. In  $\mathbb{R}^3$ , consider the linear subspaces

$$F = \{(x, y, z) \in \mathbb{R}^3, x = 0\} \text{ and } G = \{(x, y, z) \in \mathbb{R}^3, x = y\}$$

(a) Is it true that  $F \cap G = \{0_{\mathbb{R}^3}\}$ ? Justify.

The vector  $u = (0, 0, 15) \in F \cap G$ . Thus,  $F \cap G \neq \{0_{\mathbb{R}^3}\}$ .

(b) Write the definition of the set  $F + G$ .

$$F + G = \{u \in E \text{ tel que } \exists (u_1, u_2) \in F \times G \text{ tel que } u = u_1 + u_2\}$$

(c) Is vector  $u = (1, 2, 3)$  an element of  $F + G$ ? Justify.

For example, we can write  $u = (1, 2, 3) = u_1 + u_2$  with  $u_1 = (0, 1, 6) \in F$  and  $u_2 = (1, 1, -3) \in G$ . Thus,  $u \in F + G$ .

(d) Consider the decomposition that you got at question (c). Is it unique? Justify. Explain why you could know it before doing any computation.

No, the decomposition above is not unique. For example, we can also write  $u = (1, 2, 3) = v_1 + v_2$  with  $v_1 = (0, 1, 0) \in F$  and  $v_2 = (1, 1, 3) \in G$ . There is an infinite number of possibilities. We could know that from question (a), because  $F \cap G \neq \{0_{\mathbb{R}^3}\}$ .

2. Let  $E$  be a  $\mathbb{R}$ -vector space and  $\mathcal{F} = (u_1, u_2, \dots, u_n) \in E^n$  a  $n$ -vector family of  $E$ . ( $n \in \mathbb{N}^*$ )

(a) Write the mathematical definition of:  $\mathcal{F}$  is linearly independent.

$$\mathcal{F} \text{ linearly independent} \iff \forall (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n, (\lambda_1 u_1 + \dots + \lambda_n u_n = 0_E \implies \lambda_1 = \dots = \lambda_n = 0)$$

(b) Write the mathematical definition of:  $\mathcal{F}$  is a spanning family of  $E$ .

$$\mathcal{F} \text{ spanning family of } E \iff \forall u \in E, \exists (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n \text{ such that } u = \lambda_1 u_1 + \dots + \lambda_n u_n$$

(c) In  $E = \mathbb{R}^3$ , find an example of a 2-vector family which is independent. Then an example of a 3-vector family which is not independent. You don't have to justify.

- Let  $u_1 = (1, 0, 0) \in \mathbb{R}^3$  and  $u_2 = (0, 1, 0) \in \mathbb{R}^3$ . The family  $(u_1, u_2)$  is linearly independent because the two vectors are not colinear.

- Let  $\mathbb{R}^3$ ,  $v_1 = (1, 2, 3)$ ,  $v_2 = (4, 5, 6)$  and  $v_3 = (5, 7, 9)$ . The family  $(v_1, v_2, v_3)$  is linearly dependent because  $v_3 = v_1 + v_2$ .

(d) In  $E = \mathbb{R}^2$ , find an example of a spanning family of  $E$ . You don't have to justify.

The family  $(e_1 = (1, 0), e_2 = (0, 1))$  is a spanning family of  $\mathbb{R}^2$  because, for all  $u = (x, y) \in \mathbb{R}^2$ ,  $u = xe_1 + ye_2$ .