EPITA

Mathematics

Midterm exam S2

Duration: 3 hours

March 2022

Name:

First name:

Class:

MARK:

The marking system is given for a grading scale from 0 to 35.
The final mark will be re-scaled from 0 to 20.

Instructions:

- Read the whole exam before starting. **It contains 7 exercises.**
- **A 1-point penalty may be removed if the worksheet's presentation is confused.**
- Write your answers on the stapled sheets provided for answering. **Look at the frame's size before writing your redaction.**
- Documents and pocket calculators are not allowed.
- Please, do not use lead pencils for answering.

Exercise 1 (5,5 points)

1. Solve on $I =]0, +\infty[$ the differential equation (E) $xy' + \dfrac{1}{2}y = -2$.

2. Solve on $\mathbb{R}$ the differential equation (E) $2y'' + 8y' + 8y = 3e^{-2x}$

Exercise 2 (5 points)

The questions of the exercise are mutually independent.

1. Let f and g be two functions such that, as x approaches 0:

$$f(x) = o(x^3) \quad \text{and} \quad g(x) = x^2\varepsilon(x) \quad \text{with} \quad \lim_{x\to 0} \varepsilon(x) = 0$$

   (a) Can we say that, as x approaches 0, $f(x) = o(x^2)$? And that $f(x) = o(x^4)$? Justify your answers.

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   (b) Find the greatest natural number n such that we are sure to have: $f(x) - 2g(x) = o(x^n)$.

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2. Consider two functions f and g such that, as x approaches 0:

$$f(x) = 1 + x + x^2 + o(x^3) \quad \text{and} \quad g(x) = 2x + x^2 - x^3 + o(x^3)$$

   Find simple equivalents in 0 of: $f(x)$, $g(x)$ and $2xf(x) - g(x)$.

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3. Propose a Taylor expansion in 0, at the order 3, of a non-zero function h which would satisfy:

$$h(x) \sim -3x \quad \text{and} \quad h(x) + 3x \sim 5x^2$$

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4. Propose a Taylor expansion in 0, at the order 4, of a non-zero function i which would satisfy:

$$i(x) = o(x^3) \quad \text{and} \quad \lim_{x\to 0} \frac{i(x)}{x^4} = 2$$

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Exercise 3 (5 points)

In your redaction, write explicitly the basic Taylor expansions that you use.

1. Compute the Taylor expansion in 0 at the order 3 of $f(x) = \sin(2x)e^{-x}$.

2. Compute the Taylor expansion in 0 at the order 3 of $g(x) = \ln\big(1 + x + \cos(x)\big)$.

Exercise 4 (5 points)

1. Find $\lim\limits_{x \to 0} \dfrac{\sqrt{1+2x^2} - \cos\left(2x^2\right) - x^2}{e^{-x} + \sin(x) - 1}$

2. Find $\lim\limits_{x \to +\infty} \left(x \sin\left(\dfrac{1}{x}\right)\right)^{x^2}$.

Exercise 5 (6 points)

Are the following sets $\mathbb{R}$-vector spaces? Justify rigorously your answers.

1. $E = \left\{(u_n) \in \mathbb{R}^{\mathbb{N}}, \forall\, n \in \mathbb{N}, u_n \geq -1\right\}$.

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2. $F = \left\{u \in \mathbb{R}^3,\ u = \alpha\, e_1 + \beta\, e_2; (\alpha, \beta) \in \mathbb{R}^2\right\}$ where $e_1 = (1, 1, 0)$ and $e_2 = (0, 5, 3)$.

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3. $G = \{f : \mathbb{R} \longrightarrow \mathbb{R}, f(x) = o(x)$ as x approaches $0\}$

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Exercise 6 (4,5 points)

Consider a natural number $n \geq 5$ and the polynomial $P_n(X) = X^{n+1} - 2X^n + 2X^{n-1} - 2X^{n-2} + X^{n-3}$.

1. Check that 0 is a root of P. Without doing computations, find its order of multiplicity and explain why.

2. Show that 1 is a root of P. Find its order of multiplicity.

3. Assume in this question that $n = 11$. Thus, $P_{11}(X) = X^{12} - 2X^{11} + 2X^{10} - 2X^9 + X^8$. Using the previous questions, factorize P_{11} as a product of irreducible polynomials in $\mathbb{R}[X]$.

Exercise 7 (4 points)

The purpose of the exercise is to find all the polynomials P of degree 3 such that $(X-1)^2|P(X)-1$ and $(X+1)^2|P(X)+1$.

Consider a polynomial $P(X) = aX^3 + bX^2 + cX + d$ with $(a,b,c,d) \in \mathbb{R}^4$ and satisfying the hypothesis:

$$(H) \ : \ \ (X-1)^2 \mid P(X)-1 \ \text{ and } \ (X+1)^2 \mid P(X)+1$$

Let us define the two polynomials: $A(X) = P(X) - 1$ and $B(X) = P(X) + 1$.

1. Write **all** the information about A and B that can be deduced from the hypothesis (H)?

2. Deduce the values of $P(1)$, $P'(1)$, $P(-1)$ and $P'(-1)$.

3. Find all the polynomials P of degree 3 who satisfy (H).