Exercise 1 (5,5 points)

1. Solve on $I =]0, +\infty[$ the differential equation $(E)\ xy' + \frac{1}{2}y = -2$.

Homogeneous equation : $x y' + \frac{1}{2} y = 0$

$y = k e^{-\int \frac{1}{2x} dx} = k e^{-\frac{1}{2}\ln(x)} = \frac{k}{\sqrt{x}}$

Particular solution : the constant function $y : x \longmapsto -4$ is an evident solution

Conclusion: $S = \left\{ \begin{array}{l}]0,+\infty[\longrightarrow \mathbb{R} \\ x \longmapsto -4 + \frac{k}{\sqrt{x}} \end{array}, k \in \mathbb{R} \right\}$

2. Solve on $\mathbb{R}$ the differential equation $(E)\ 2y'' + 8y' + 8y = 3e^{-2x}$

Homogeneous equation : $y'' + 4y' + 4y = 0$

. The characteristic equation is $X^2 + 4X + 4 = 0$

$\Leftrightarrow \cancel{X+2} \ (X+2)^2 = 0$

one double root : $X = -2$.

Thus, $y(x) = (k_1 x + k_2) e^{-2x}$, $(k_1, k_2) \in \mathbb{R}^2$

Particular solution : we search y_P in the form $y_P = P(x) e^{-2x}$

Then $y_P' = (P' - 2P) e^{-2x}$, $y_P'' = (P'' - 4P' + 4P) e^{-2x}$

$y_P \in S \Leftrightarrow 2(P'' - 4P' + 4P) e^{-2x} + 8(P' - 2P) e^{-2x} + 8P e^{-2x} = 3e^{-2x}$

$\Leftrightarrow e^{-2x}(2P'' + 0P' + 0P) = 3e^{-2x}$

$\Leftrightarrow P'' = \frac{3}{2}$

Let $P' = \frac{3}{2}x$, $P'' = \frac{3}{4}x^2$, then $y_P(x) = \frac{3}{4}x^2 e^{-2x}$

$S = \left\{ \begin{array}{l} \mathbb{R} \longrightarrow \mathbb{R} \\ x \longmapsto \left(\frac{3}{4}x^2 + k_1 x + k_2\right) e^{-2x} \end{array}, (k_1, k_2) \in \mathbb{R}^2 \right\}$

Exercise 2 (5 points)

The questions of the exercise are mutually independent.

1. Let f and g be two functions such that, as x approaches 0:

$$f(x) = o(x^3) \text{ and } g(x) = x^2\varepsilon(x) \text{ with } \lim_{x\to 0}\varepsilon(x) = 0$$

(a) Can we say that, as x approaches 0, $f(x) = o(x^2)$? And that $f(x) = o(x^4)$? Justify your answers.

$f(x)$ has the form $x^3\,\varepsilon_2(x)$ with $\varepsilon_2(x) \longrightarrow 0$

Thus, $\frac{f(x)}{x^2} = \frac{x^3\varepsilon_2(x)}{x^2} = x\,\varepsilon_2(x) \longrightarrow 0 \Rightarrow f(x) = o(x^2)$

But $\frac{f(x)}{x^4} = \frac{\varepsilon_2(x)}{x}$, we cannot know whether $f(x) = o(x^4)$ or not.

(b) Find the greatest natural number n such that we are sure to have: $f(x) - 2g(x) = o(x^n)$.

$f(x) - 2g(x) = o(x^2)$ because $f(x) - 2g(x) = x^3\,\varepsilon_2(x) - x^2\varepsilon(x)$
$= x^2\underbrace{\left(x\,\varepsilon_2(x) - \varepsilon(x)\right)}_{\longrightarrow 0}$

But we're not sure that $f(x) - 2g(x) = o(x^3)$:
$\frac{f(x) - 2g(x)}{x^3} = \frac{x^3\,\varepsilon_2(x) - x^2\varepsilon(x)}{x^3} = \varepsilon_2(x) - \frac{\varepsilon(x)}{x}$ maybe does not tend to 0.

2. Consider two functions f and g such that, as x approaches 0:

$$f(x) = 1 + x + x^2 + o(x^3) \text{ and } g(x) = 2x + x^2 - x^3 + o(x^3)$$

Find simple equivalents in 0 of: $f(x)$, $g(x)$ and $2xf(x) - g(x)$.

$f(x) = 1 + o(1) \Rightarrow f(x) \sim 1$
$g(x) = 2x + o(x) \Rightarrow g(x) \sim 2x$
$2xf(x) - g(x) = 2x + 2x^2 + 2x^3 + o(x^4) - 2x - x^2 + x^3 + o(x^3) = x^2 + o(x^2) \sim x^2$

3. Propose a Taylor expansion in 0, at the order 3, of a non-zero function h which would satisfy:

$$h(x) \sim -3x \text{ and } h(x) + 3x \sim 5x^2$$

Let $h(x) = -3x + 5x^2 + 37x^3 + o(x^3)$
Then $h(x) = -3x + o(x) \Rightarrow h(x) \sim -3x$ and $h(x) + 3x = 5x^2 + o(x^2) \sim 5x^2$

4. Propose a Taylor expansion in 0, at the order 4, of a non-zero function i which would satisfy:

$$i(x) = o(x^3) \text{ and } \lim_{x\to 0}\frac{i(x)}{x^4} = 2$$

Let $i(x) = 2x^4 + o(x^4)$

Exercise 3 (5 points)

In your redaction, write explicitly the basic Taylor expansions that you use.

1. Compute the Taylor expansion in 0 at the order 3 of $f(x) = \sin(2x)e^{-x}$.

$$\sin(2x)e^{-x} = \left(2x - \frac{(2x)^3}{3!} + o(x^3)\right)\left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + o(x^3)\right)$$

$$= \left(2x - \frac{4}{3}x^3 + o(x^3)\right)\left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + o(x^3)\right)$$

$$= 2x - 2x^2 + x^3 + o(x^3)$$
$$- \frac{4}{3}x^3$$

$$= 2x - 2x^2 - \frac{1}{3}x^3 + o(x^3)$$

2. Compute the Taylor expansion in 0 at the order 3 of $g(x) = \ln(1 + x + \cos(x))$.

$$\ln(1 + x + \cos(x)) = \ln\left(1 + x + 1 - \frac{x^2}{2} + o(x^3)\right)$$

$$= \ln\left(2 + x - \frac{x^2}{2} + o(x^3)\right)$$

$$= \ln\left(2\left(1 + \frac{x}{2} - \frac{x^2}{4} + o(x^3)\right)\right)$$

$$= \ln(2) + \ln\left(1 + \frac{x}{2} - \frac{x^2}{4} + o(x^3)\right)$$

$$= \ln(2) + \left(\frac{x}{2} - \frac{x^2}{4} + o(x^3)\right) - \frac{1}{2}\left(\frac{x}{2} - \frac{x^2}{4} + o(x^3)\right)^2$$
$$+ \frac{1}{3}\left(\frac{x}{2} - \frac{x^2}{4} + o(x^3)\right)^3 + o(x^3)$$

$$= \ln(2) + \left(\frac{x}{2} - \frac{x^2}{4} + o(x^3)\right) - \frac{1}{2}\left(\frac{x^2}{4} - \frac{x^3}{4} + o(x^3)\right)$$
$$+ \frac{1}{3}\left(\frac{x^3}{8} + o(x^3)\right)$$

$$= \ln(2) + \frac{x}{2} - \frac{3}{8}x^2 + \frac{x^3}{6} + o(x^3)$$

Exercise 4 (5 points)

1. Find $\lim_{x\to 0} \dfrac{\sqrt{1+2x^2}-\cos(2x^2)-x^2}{e^{-x}+\sin(x)-1}$

The denominator is

$$D(x) = \left(1 - x + \frac{x^2}{2} + o(x^2)\right) + \left(x + o(x^2)\right) - 1$$
$$= \frac{x^2}{2} + o(x^2) \sim \frac{x^2}{2}$$

The numerator is

$$N(x) = 1 + \frac{1}{2}(2x^2) + \frac{\frac{1}{2}(-\frac{1}{2})}{2}(2x^2)^2 + o(x^4) - \left(1 - \frac{(2x^2)^2}{2} + o(x^4)\right) - x^2$$
$$= 1 + x^2 - \frac{x^4}{2} + o(x^4) - 1 + 2x^4 + o(x^4) - x^2$$
$$= \frac{3}{2}x^4 + o(x^4) \sim \frac{3}{2}x^4$$

Thus, $\dfrac{N(x)}{D(x)} \sim \dfrac{3/2\, x^4}{x^2/2} = 3x^2.$

Since $3x^2 \longrightarrow 0$, $\lim_{x\to 0} \dfrac{N(x)}{D(x)} = 0$

2. Find $\lim_{x\to+\infty} \left(x\sin\left(\frac{1}{x}\right)\right)^{x^2}$.

$$\left(x\sin\left(\tfrac{1}{x}\right)\right)^{x^2} = e^{x^2 \ln\left(x\sin\left(\frac{1}{x}\right)\right)}$$
$$= e^{x^2 \ln\left(x\left(\frac{1}{x} - \frac{1}{6x^3} + o\left(\frac{1}{x^3}\right)\right)\right)}$$
$$= e^{x^2 \ln\left(1 - \frac{1}{6x^2} + o\left(\frac{1}{x^2}\right)\right)}$$
$$= e^{x^2\left(-\frac{1}{6x^2} + o\left(\frac{1}{x^2}\right)\right)}$$
$$= e^{-\frac{1}{6} + o(1)} \xrightarrow[x\to+\infty]{} e^{-1/6}$$

Exercise 5 (6 points)

Are the following sets $\mathbb{R}$-vector spaces? Justify rigorously your answers.

1. $E = \{(u_n) \in \mathbb{R}^{\mathbb{N}}, \forall n \in \mathbb{N}, u_n \geq -1\}$.

$E \subset \mathbb{R}^{\mathbb{N}}$.

But E is not a linear subspace of $\mathbb{R}^{\mathbb{N}}$.

If (u_n) is the sequence defined for all $n \in \mathbb{N}$ by $u_n = 2$,

then $(u_n) \in E$ because $2 \geqslant -1$.

But $-(u_n) \notin E$ because $-2 < -1$.

Thus, E is not a $\mathbb{R}$-vs.

2. $F = \{u \in \mathbb{R}^3,\ u = \alpha\, e_1 + \beta\, e_2; (\alpha, \beta) \in \mathbb{R}^2\}$ where $e_1 = (1, 1, 0)$ and $e_2 = (0, 5, 3)$.

$F \subset \mathbb{R}^3$. F is a linear subspace of $\mathbb{R}^3$. Indeed:

* $0_{\mathbb{R}^3} = 0 e_1 + 0 e_2 \Rightarrow 0_{\mathbb{R}^3} \in F$.

* Let $(u, v) \in F^2$ and $\lambda \in \mathbb{R}$.

$\exists (\alpha_1, \beta_1) \in \mathbb{R}^2$, $u = \alpha_1 e_1 + \beta_1 e_2$, $\exists (\alpha_2, \beta_2) \in \mathbb{R}^2$, $v = \alpha_2 e_1 + \beta_2 e_2$

Then $\lambda u + v = \lambda(\alpha_1 e_1 + \beta_1 e_2) + (\alpha_2 e_1 + \beta_2 e_2) = (\lambda\alpha_1 + \alpha_2) e_1 + (\lambda\beta_1 + \beta_2) e_2$

so $\lambda u + v \in F$.

3. $G = \{f : \mathbb{R} \longrightarrow \mathbb{R}, f(x) = o(x) \text{ as } x \text{ approaches } 0\}$

$G \subset \mathbb{R}^{\mathbb{R}}$ and G is a linear subspace of $\mathbb{R}^{\mathbb{R}}$

* Let Z be the zero function. Then $Z \in G$ since $\lim\limits_{x \to 0} \frac{Z(x)}{x} = 0$

* Let $(f, g) \in G^2$ and $\lambda \in \mathbb{R}$.

Then $\lim\limits_{x \to 0} \frac{\lambda f(x) + g(x)}{x} = \lambda \lim\limits_{x \to 0} \frac{f(x)}{x} + \lim\limits_{x \to 0} \frac{g(x)}{x} = 0$

So $\lambda f(x) + g(x) = o(x) \Rightarrow \lambda f + g \in G$

Exercise 6 (4,5 points)

Consider a natural number $n \geq 5$ and the polynomial $P_n(X) = X^{n+1} - 2X^n + 2X^{n-1} - 2X^{n-2} + X^{n-3}$.

1. Check that 0 is a root of P. Without doing computations, find its order of multiplicity and explain why.

$P_n(X) = (X-0)^{n-3}(X^4 - 2X^3 + 2X^2 - 2X + 1)$

so $X^{n-3} \mid P_n$ and $X \nmid (X^4 - 2X^3 + 2X^2 - 2X + 1)$ ∴ $m(0) = n-3$.

2. Show that 1 is a root of P. Find its order of multiplicity.

Let $A = X^4 - 2X^3 + 2X^2 - 2X + 1$

$A(1) = 1 - 2 + 2 - 2 + 1 = 0$

$A' = 4X^3 - 6X^2 + 4X - 2 \Rightarrow A'(1) = 4 - 6 + 4 - 2 = 0$

$A'' = 12X^2 - 12X + 4 \Rightarrow A''(1) = 4 \neq 0$

1 is a root of multiplicity 2 of A.
It is a root of multiplicity 2 of P_n
(since $X^{n-3} \wedge (X-1) = 1$)

3. Assume in this question that $n = 11$. Thus, $P_{11}(X) = X^{12} - 2X^{11} + 2X^{10} - 2X^9 + X^8$. Using the previous questions, factorize P_{11} as a product of irreducible polynomials in $\mathbb{R}[X]$.

$P_{11} = X^8(X^4 - 2X^3 + 2X^2 - 2X + 1) = X^8 \times A$

Euclidean division of A by $(X-1)^2$: we get
$A(X) = (X-1)^2(X^2+1)$

Thus, $P_{11} = X^8 (X-1)^2 (X^2+1)$

(and X^2+1 is irreducible)

Exercise 7 (4 points)

The purpose of the exercise is to find all the polynomials P of degree 3 such that $(X-1)^2|P(X)-1$ and $(X+1)^2|P(X)+1$.
Consider a polynomial $P(X) = aX^3 + bX^2 + cX + d$ with $(a,b,c,d) \in \mathbb{R}^4$ and satisfying the hypothesis:

$$(H)\ :\ (X-1)^2 \mid P(X)-1 \ \text{ and } \ (X+1)^2 \mid P(X)+1$$

Let us define the two polynomials: $A(X) = P(X) - 1$ and $B(X) = P(X) + 1$.

1. Write **all** the information about A and B that can be deduced from the hypothesis (H)?

$A(1) = A'(1) = 0$ and $B(-1) = B'(-1) = 0$

2. Deduce the values of $P(1)$, $P'(1)$, $P(-1)$ and $P'(-1)$.

$P(1) = 1$ $\quad P(-1) = -1$
$P'(1) = 0$ $\quad P'(-1) = 0$

3. Find all the polynomials P of degree 3 who satisfy (H).

Let $P = aX^3 + bX^2 + cX + d$. Then $P' = 3aX^2 + 2bX + c$

$$\begin{cases} a + b + c + d = 1 & (P(1)=1) \\ 3a + 2b + c = 0 & (P'(1)=0) \\ -a + b - c + d = -1 & (P(-1)=-1) \\ 3a - 2b + c = 0 & (P'(-1)=0) \end{cases}$$

$$\iff \begin{cases} a + b + c + d = 1 & \\ b + d = 0 & (Eq\,1 + Eq\,3) \\ 3a + 2b + c = 0 & (Eq\,2) \\ 4b = 0 & (Eq\,2 - Eq\,4) \end{cases}$$

$\iff b = 0$, $d = -b = 0$, $c = -3a$ (from Eq 3)

We inject in Eq 1: $-2a = 1 \Rightarrow a = -\frac{1}{2}$ and $c = \frac{3}{2}$

We get: $P = -\frac{X^3}{2} + \frac{3}{2}X$