Midterm exam n°2

Duration : three hours
Documents and calculators not allowed

Name : **First name :** **Class :**

Instructions :

— You have to reply directly on the given sheets.

— ***No sheet other that the stapled ones provided for answers will be corrected.***

— Answers written using lead pencils will be ignored.

— Any student not respecting these instructions will be awarded a mark of 00/ 20.

Exercise 1 (4,5 points)

1. Using two consecutive integrations by parts, calculate $I = \int_1^e \sin\left(\ln(x)\right) \mathrm{d}x$.

2. Using an integration by parts, calculate $J = \int_0^1 \arctan(x)\,\mathrm{d}x$.

3. Using the substitution $u = \sqrt{x}$ then an integration by parts, calculate $K = \int_0^{\pi^2} \cos\left(\sqrt{x}\right)\mathrm{d}x$.

Exercise 2 (3 points)

Let (u_n) and (v_n) be two strictly positive numerical sequences such that for every $n \in \mathbb{N}$,

$$\frac{u_{n+1}}{u_n} \leqslant \frac{v_{n+1}}{v_n}$$

1. Prove that if $v_n \xrightarrow[n\to+\infty]{} 0$ then $u_n \xrightarrow[n\to+\infty]{} 0$.

2. Prove that if $u_n \xrightarrow[n\to+\infty]{} +\infty$ then $v_n \xrightarrow[n\to+\infty]{} +\infty$.

Exercise 3 (3 points)

Circle the letters corresponding to the true statements only.
Remark that unlike usually, ***wrong answers do not award negative points***!

a. Let (u_n) be a sequence of real numbers, and $\ell \in \mathbb{R}$. Then the assertion « if (u_n) converges towards ℓ then, for every $n \in \mathbb{N}$, $u_n \leqslant \ell$ » is equivalent to the assertion « if there exists $n \in \mathbb{N}$ such that $u_n > \ell$, then (u_n) does not converge towards ℓ ».

b. If (u_n) is a nonzero geometric sequence with common ratio $q \in \mathbb{R}^*$, then $\left(\frac{1}{u_n}\right)$ is a geometric sequence with common ratio $\frac{1}{q}$.

c. If (u_n) is a bounded numerical sequence, there exists a subsequence of (u_n) that is convergent.

d. Let (u_n) be a numerical sequence. Then (u_{6n}) is a subsequence of (u_{2n}).

e. Let (u_n) be a numerical sequence. Then $(u_{3\cdot 2^{n+1}})$ is a subsequence of (u_{6n}).

f. None of the above.

Exercise 4 (3 points)

Let (u_n) and (v_n) be defined for every $n \in \mathbb{N}$ by $u_n = \sum_{k=0}^{2n+1} \frac{(-1)^k}{(2k)!}$ and $v_n = u_n + \frac{1}{(4n+4)!}$.

Prove that (u_n) and (v_n) are adjacent sequences.

Exercise 5 (2 points)

Let $(u_n)_{n\in\mathbb{N}^*}$ be defined for every $n \in \mathbb{N}$ by $u_n = \dfrac{\ln(n!)}{n^2}$.

1. Let $n \in \mathbb{N}^*$. Show (without using a proof by induction) that $\ln(n!) \leqslant n\ln(n)$.

2. Deduce the limit of the sequence $(u_n)_{n\in\mathbb{N}^*}$.

Exercise 6 (5,5 points)

Let (u_n) be the numerical sequence defined for every $n \in \mathbb{N}$ by $u_n = \displaystyle\sum_{k=0}^{n} \frac{1}{k!}$.

1. Let $n \in \mathbb{N}^*$ and $q \in \mathbb{R}\backslash\{1\}$. What is the sum $\displaystyle\sum_{k=1}^{n} q^{k-1} = 1 + q + q^2 + \cdots + q^{n-1}$ equal to?

2. Let $n \in \mathbb{N}^*$. Using the previous question, show (without using a proof by induction) that $\displaystyle\sum_{k=1}^{n} \frac{1}{2^{k-1}} = 2 - \frac{1}{2^{n-1}}$.

3. Let $k \in \mathbb{N}$ such $k \geqslant 2$. Show (without induction) that $\dfrac{1}{k!} = \dfrac{1}{2 \times 3 \times \cdots \times k} \leqslant \dfrac{1}{2^{k-1}}$.

   Check that this inequality is still true when $k = 1$.

4. Prove that (u_n) is increasing.

5. Using questions 2 and 3, show that for every $n \in \mathbb{N}$, $u_n \leqslant 3$.

6. Is the sequence (u_n) convergent ? Justify your answer.