Correction of midterm exam n°2

Exercise 1 (4,5 points)

1. Using an integration by parts, by setting $u(x) = \sin\big(\ln(x)\big)$ and $v'(x) = 1$, we get :

$$\begin{aligned} I &= \Big[x\sin\big(\ln(x)\big)\Big]_1^e - \int_1^e \cos\big(\ln(x)\big)\,\mathrm{d}x \\ &= e\sin(1) - \int_1^e \cos\big(\ln(x)\big)\,\mathrm{d}x \end{aligned}$$

Using a second integration by parts, by setting $u(x) = \cos\big(\ln(x)\big)$ and $v'(x) = 1$, we get :

$$I = e\sin(1) - \left(\Big[x\cos\big(\ln(x)\big)\Big]_1^e + I\right)$$

thus $2I = 1 + e\big(\sin(1) - \cos(1)\big)$, that is to say $I = \dfrac{1}{2}\Big(1 + e\big(\sin(1) - \cos(1)\big)\Big)$.

2. Using an integration by parts, by setting $u(x) = \arctan(x)$ and $v'(x) = 1$, we get :

$$\begin{aligned} J &= \big[x\arctan(x)\big]_0^1 - \int_0^1 \frac{x}{1+x^2}\mathrm{d}x \\ &= \frac{\pi}{4} - \frac{1}{2}\big[\ln(1+x^2)\big]_0^1 \\ &= \frac{\pi}{4} - \frac{1}{2}\ln(2) \end{aligned}$$

3. Using the substitution $u = \sqrt{x}$, $x = u^2$ thus $\mathrm{d}x = 2u\mathrm{d}u$.

We get : $K = 2\displaystyle\int_0^\pi u\cos(u)\,\mathrm{d}u$.

Using an integration by parts, by setting $\varphi(u) = u$ and $\psi'(u) = \cos(u)$, we get :

$$K = 2\left(\big[u\sin(u)\big]_0^\pi - \int_0^\pi \sin(u)\,\mathrm{d}u\right) = -2\int_0^\pi \sin(u)\,\mathrm{d}u$$

Thus $K = -4$.

Exercise 2 (3 points)

1. From the hypothesis we deduce that $\begin{cases} \dfrac{u_n}{u_{n-1}} \leqslant \dfrac{v_n}{v_{n-1}} \\ \vdots \qquad\quad \vdots \\ \dfrac{u_1}{u_0} \leqslant \dfrac{v_1}{v_0} \end{cases}$

By multiplying these inequalities of positive numbers, we get :

$$\frac{u_n}{u_{n-1}} \cdot \frac{u_{n-1}}{u_{n-2}} \cdot \ \cdots \ \cdot \frac{u_1}{u_0} \leqslant \frac{v_n}{v_{n-1}} \cdot \frac{v_{n-1}}{v_{n-2}} \cdot \ \cdots \ \cdot \frac{v_1}{v_0}$$

from which we deduce that $\dfrac{u_n}{u_0} \leqslant \dfrac{v_n}{v_0}$, that is to say $0 < u_n \leqslant \dfrac{u_0}{v_0}\,v_n$.

Therefore, using the squeeze theorem (sandwich theorem), if $v_n \xrightarrow[n\to+\infty]{} 0$ then $u_n \xrightarrow[n\to+\infty]{} 0$.

2. Similarly, $v_n \geqslant \dfrac{v_0}{u_0}u_n$.

Thus, still using the squeeze theorem, if $u_n \xrightarrow[n\to+\infty]{} +\infty$ then $v_n \xrightarrow[n\to+\infty]{} +\infty$.

Exercise 3 (3 points)

(a.) Let (u_n) be a sequence of real numbers, and $\ell \in \mathbb{R}$. Then the assertion « if (u_n) converges towards ℓ then, for every $n \in \mathbb{N}$, $u_n \leqslant \ell$ » is equivalent to the assertion « if there exists $n \in \mathbb{N}$ such that $u_n > \ell$, then (u_n) does not converge towards ℓ ».

(b.) If (u_n) is a nonzero geometric sequence with common ratio $q \in \mathbb{R}^*$, then $\left(\frac{1}{u_n}\right)$ is a geometric sequence with common ratio $\frac{1}{q}$.

(c.) If (u_n) is a bounded numerical sequence, there exists a subsequence of (u_n) that is convergent.

(d.) Let (u_n) be a numerical sequence. Then (u_{6n}) is a subsequence of (u_{2n}).

(e.) Let (u_n) be a numerical sequence. Then $(u_{3\cdot 2^{n+1}})$ is a subsequence of (u_{6n}).

f. none of the above

Exercise 4 (3 points)

Let us prove that (u_n) is (strictly) increasing.

$$u_{n+1} - u_n = \sum_{k=0}^{2n+3} \frac{(-1)^k}{(2k)!} - \sum_{k=0}^{2n+1} \frac{(-1)^k}{(2k)!} = \frac{(-1)^{2n+2}}{(2(2n+2))!} + \frac{(-1)^{2n+3}}{(2(2n+3))!} = \frac{1}{(4n+4)!} - \frac{1}{(4n+6)!}$$

Yet

$$(4n+6)! > (4n+4)!$$

hence

$$\frac{1}{(4n+6)!} < \frac{1}{(4n+4)!}$$

Therefore $u_{n+1} - u_n > 0$, thus (u_n) is strictly increasing.

Let us prove that (v_n) is (strictly) decreasing.

$$v_{n+1} - v_n = u_{n+1} + \frac{1}{(4(n+1)+4)!} - u_n - \frac{1}{(4n+4)!}$$

By reusing the already calculated expression of $u_{n+1} - u_n$, we get :

$$v_{n+1} - v_n = \frac{1}{(4n+4)!} - \frac{1}{(4n+6)!} + \frac{1}{(4n+8)!} - \frac{1}{(4n+4)!} = \frac{1}{(4n+8)!} - \frac{1}{(4n+6)!}$$

Yet

$$(4n+8)! > (4n+6)!$$

hence

$$\frac{1}{(4n+8)!} < \frac{1}{(4n+6)!}$$

Therefore $v_{n+1} - v_n < 0$ thus (v_n) is strictly decreasing.

Eventually, $v_n - u_n = \frac{1}{(4n+4)!} \longrightarrow 0$ thus (u_n) and (v_n) are adjacent sequences.

Exercise 5 (2 points)

1. $\ln(n!) = \ln(1) + \cdots + \ln(n) = \sum_{k=1}^{n} \ln(k) \leqslant n\ln(n)$ as the function ln is increasing.
2. Thus, for every $n \in \mathbb{N}^*$, $0 \leqslant u_n \leqslant \frac{\ln(n)}{n}$.

   Using the squeeze theorem together with a result of compared growth, $(u_n)_{n\in\mathbb{N}^*}$ converges towards 0.

Exercise 6 (5,5 points)

1. As $q \neq 1$, $\displaystyle\sum_{k=1}^{n} q^{k-1} = \frac{1-q^n}{1-q}$

2. Using the previous question,

$$\sum_{k=1}^{n} \frac{1}{2^{k-1}} = \sum_{k=1}^{n} \left(\frac{1}{2}\right)^{k-1} = \frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}} = 2\left(1-\frac{1}{2^n}\right)$$

   that is to say

$$\sum_{k=1}^{n} \frac{1}{2^{k-1}} = 2 - \frac{1}{2^{n-1}}$$

3. Let $k \geqslant 2$. Then : $\begin{cases} 2 \geqslant 2 \\ 3 \geqslant 2 \\ \vdots \\ k \geqslant 2 \end{cases}$ hence $\begin{cases} \frac{1}{2} \leqslant \frac{1}{2} \\ \frac{1}{3} \leqslant \frac{1}{2} \\ \vdots \\ \frac{1}{k} \leqslant \frac{1}{2} \end{cases}$

   Thus, $\displaystyle\frac{1}{k!} = \frac{1}{2 \times 3 \times \cdots \times k} = \frac{1}{2} \times \frac{1}{3} \times \cdots \times \frac{1}{k} \leqslant \frac{1}{2^{k-1}}$

   This property still holds when $k = 1$ as $\displaystyle\frac{1}{1!} = 1 \leqslant \frac{1}{2^0} = 1$.

4. Let $n \in \mathbb{N}$. Then $u_{n+1} - u_n = \displaystyle\frac{1}{(n+1)!} \geqslant 0$ thus (u_n) is increasing.

5. Using questions 2 and 3,

$$u_n = 1 + \sum_{k=1}^{n} \frac{1}{k!} \leqslant 1 + \sum_{k=1}^{n} \frac{1}{2^{k-1}} \leqslant 1 + 2 - \frac{1}{2^{n-1}} \leqslant 3$$

6. (u_n) is an increasing numerical sequence that is bounded above, thus it is convergent.