EPITA

Mathematics

Final exam S2

Duration: 3 hours

May 2022

Name:

First name:

Class:

Mark:

The marking system is given for a grading scale from 0 to 40.
The final mark will be re-scaled from 0 to 20.

Instructions:

- Read the whole exam before starting. **It contains 7 exercises.**
- **The quality of your redaction will be accounted in your mark.**
- Write your answers on the stapled sheets provided for answering. **Look at the frame's size before writing your redaction.**
- A 1-point penalty may be removed if the paper's presentation is confused.
- Documents and pocket calculators are not allowed.
- Please, do not use lead pencils for answering.

Exercise 1 (6.5 points)

1. Recall the Taylor expansions in 0, at the order 4, of e^u, $\cos(u)$, $\sin(u)$, $\ln(1+u)$ and $\sqrt{1+u}$.

2. Find the Taylor expansion in 0 at the order 4 of $f(x) = \ln(1-2x)\sin\left(\frac{x}{2}\right)$.

3. Find the Taylor expansion in 0 at the order 4 of $g(x) = \sqrt{1+\cos(2x)}$.

Exercise 2 (4 points)

1. Compute $\lim_{x\to 0} \dfrac{\cos(2x^2)-1}{e^x - e^{-x} - 2x}$

2. Compute $\lim_{x\to+\infty} x^2\left(e^{\frac{1}{x^2}} - \cos\left(\dfrac{1}{x}\right)\right)$.

Exercise 3 (5 points)

Consider the endomorphism f of $\mathbb{R}^3$ defined by its matrix in the standard basis (as input and output basis): $A = \begin{pmatrix} 2 & 2 & 3 \\ 4 & 3 & 5 \\ -1 & -2 & -2 \end{pmatrix}$.

1. Find the inverse of A.

2. Is the endomorphism f bijective? Explain why. If it is, give an expression of the mapping f^{-1}.

3. Assume that A is the matrix of another linear map $g \in \mathcal{L}\left(\mathbb{R}_2[X], \mathbb{R}^3\right)$ in the standard bases as input and output bases. Give an expression of the mapping g.

Exercise 4 (5 points)

The two questions of this exercise are mutually independent.

1. In the vector space $E = \mathbb{R}_2[X]$, consider the families:

$$\mathcal{F}_1 = \left(P_1 = 1, P_2 = X^2 + X, P_3 = -X^2 - X - 2\right) \quad \text{and} \quad \mathcal{F}_2 = \left(Q_1 = 1, Q_2 = X^2 + X + 1, Q_3 = X^2 - X\right)$$

For each family, say whether it is a basis of E or not. Justify rigorously your answer. If it is, find the coordinates of $P = -X^2 + 7X + 5$ in this basis.

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2. In the vector space $\mathbb{R}^3$, consider the vectors $u = (2, 3, -1)$ and $v = (1, -1, -2)$.

   (a) Is the family (u, v) a basis of $G = \mathrm{Span}(u, v)$? Justify your answer.

   (b) Find $a \in \mathbb{R}$ such that $w = (-15, 5, a) \in G$.

Exercise 5 (8 points)

Consider the linear map $f : \left\{ \begin{array}{lcl} \mathbb{R}_2[X] & \longrightarrow & \mathbb{R}^2 \\ P & \longmapsto & (P(0), P'(1)) \end{array} \right.$

1. Find a basis of the kernel of f. Deduce its dimension.

2. Write the global statement of the rank-nullity theorem. In the case of this particular linear map, find the dimension of $\mathrm{Im}(f)$.

3. Is f injective? Is f surjective?

4. Find the matrix of f in the standard bases as input and output bases.

5. Find the matrix of f in the standard basis as input basis and $\big((2,1),(-1,-1)\big)$ as output basis.

Exercise 6 (8 points)

Let $f \in \mathcal{L}(\mathbb{R}^3)$ be defined by its matrix in the standard basis: $A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ 0 & -1 & -1 \end{pmatrix}$.

The three column vectors of A are denoted by C_1, C_2 and C_3.

1. Is the family (C_1, C_2, C_3) linearly independent? If it is not, extract a maximal independent family.

2. Deduce a basis $\mathcal{B}_1$ of $\mathrm{Im}(f)$ and a basis $\mathcal{B}_2$ of $\mathrm{Ker}(f)$. Give their dimensions.

3. Show that $\mathrm{Im}(f) \oplus \mathrm{Ker}(f) = \mathbb{R}^3$. We build a basis $\mathcal{B}$ of $\mathbb{R}^3$ by concatenation of the families $\mathcal{B}_1$ and $\mathcal{B}_2$.

4. Compute A^2 and deduce a property about f. How such a linear map is called?

5. Using questions 3. and 4., give the image by f of an arbitrary vector in $\mathrm{Im}(f)$. Then of an arbitrary vector in $\mathbb{R}^3$.

6. With no computations, using question 5, find the matrix of f in basis $\mathcal{B}$ as input and output basis.

Exercise 7 (3.5 points)

Let $E = \mathbb{R}^2$, $F = \text{Span}\big((1,1)\big)$ and $G = \text{Span}\big((1,-1)\big)$. We accept without proof that $F \oplus G = E$. Thus, $\forall\, u \in E$, $\exists!(v,w) \in F \times G$ such that $u = v + w$.

Consider the linear map $s : \begin{cases} E & \longrightarrow & E \\ u & \longmapsto & v - w \end{cases}$

1. For $u = (1,3) \in \mathbb{R}^2$, find v and w. Deduce $s(u)$.

2. Draw F and G in $\mathbb{R}^2$, then draw the vector $u = (1,3)$. Explain graphically how we get v, w and $s(u)$ (draw these vectors too). How could we call the endomorphism s?