EPITA

Mathematics S2

Final exam

June 2021

Duration: 3 hours

Name:

First name:

Class:

MARK:

The marking system is given for a grading scale from 0 to 30.
The final mark will be re-scaled from 0 to 20.

Instructions:

— Documents and pocket calculators are not allowed.
— Write your answers on the stapled sheets provided for answering. No other sheet will be corrected.
— Please, do not use lead pencils for answering.

For the whole exam, each answer must be accurately justified.

Exercise 1 (4,5 points)

Let $E = \mathbb{R}_2[X]$.

a. For each of the families below, say whether it is a basis of E or not.

$\mathcal{F}_1 = \left\{P_1(X) = 2X + 1,\ P_2(X) = X^2 + X + 2,\ P_3(X) = X^2 + X + 1,\ P_4(X) = 2X^2 + 3\right\}$

$\mathcal{F}_2 = \left\{Q_1(X) = X^2 + 2,\ Q_2(X) = X^2 + 4X,\ Q_3(X) = X^2 + 3X + 2\right\}$

$\mathcal{F}_3 = \left\{R_1(X) = -X^2 + 2,\ R_2(X) = X^2 - 4X,\ R_3(X) = X^2 - 2X - 1\right\}$

b. Find the coordinates of the polynomial $P(X) = 2X^2 + X + 8$ in each basis that you have found at question a.

Exercise 2 (2,5 points)

Let $(a,b,c) \in \mathbb{R}^3$ and $\varphi : \left\{ \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & ax^2 + bx + c \end{array} \right.$

Determine necessary and sufficient conditions on (a,b,c) for φ to be a linear map. Prove your answer.

Exercise 3 (5 points)

Consider the vector space $\mathbb{R}^{\mathbb{R}}$ containing all the functions from $\mathbb{R}$ to $\mathbb{R}$, and the following subsets of $\mathbb{R}^{\mathbb{R}}$: $\mathcal{E}$ the set of even functions ($\forall x \in \mathbb{R},\ f(-x) = f(x)$) and $\mathcal{O}$ the set of odd functions ($\forall x \in \mathbb{R},\ f(-x) = -f(x)$).

a. Show that $\mathcal{E}$ is a linear subspace of $\mathbb{R}^{\mathbb{R}}$. (We accept without proof that $\mathcal{O}$ is a linear subspace of $\mathbb{R}^{\mathbb{R}}$ too)

b. Using an example, show that $\mathcal{E} \cup \mathcal{O} \neq \mathbb{R}^{\mathbb{R}}$

c. Let $f \in \mathbb{R}^{\mathbb{R}}$ and consider the two following functions:

$$f_e : \begin{cases} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \dfrac{f(x) + f(-x)}{2} \end{cases} \quad \text{and } f_o : \begin{cases} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \dfrac{f(x) - f(-x)}{2} \end{cases}.$$

Show that f_e is even and that f_o is odd. Compute $f_e + f_o$.

d. Show that $\mathcal{E}$ and $\mathcal{O}$ are supplementary in $\mathbb{R}^{\mathbb{R}}$.

Exercise 4 (4 points)

Consider the linear map $f : \begin{cases} \mathbb{R}^3 & \longrightarrow & \mathbb{R}[X] \\ (a,b,c) & \longmapsto & (a+b)X^4 + (2a-c)X^2 + (a+b+c) \end{cases}$

a. Determine $\operatorname{Ker} f$, $\dim \operatorname{Ker} f$ and $\dim \operatorname{Im} f$.

b. Is f injective?

c. Is f surjective?

Exercise 5 (4 points)

Let $g \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^4)$ be the linear map associated to the matrix $\begin{pmatrix} 1 & 0 & 1 \\ 2 & 3 & 0 \\ 1 & 0 & 0 \\ 2 & 3 & 1 \end{pmatrix}$ in the standard bases as input and output bases.

Let $\mathcal{B}_3 = \{e_1, e_2, e_3\}$ be the standard basis of $\mathbb{R}^3$ and $\mathcal{B}_4$ the standard basis of $\mathbb{R}^4$.
Consider the two bases: $\mathcal{D}_3 = \{u_1, u_2, u_3\}$ basis of $\mathbb{R}^3$ where $u_1 = (1, -1, 0),\ u_2 = (1, 0, 1),\ u_3 = (0, 1, -1)$ and $\mathcal{D}_4 = \{v_1, v_2, v_3, v_4\}$ basis of $\mathbb{R}^4$ where $v_1 = (0, 0, 0, 1),\ v_2 = (1, 0, 0, 1),\ v_3 = (1, 2, 1, 2),\ v_4 = (0, 3, 0, 3)$.

a. What are the images of the vectors of $\mathcal{B}_3$ by g?

b. Let $(x, y, z) \in \mathbb{R}^3$, compute $g\big((x, y, z)\big)$.

c. Determine the matrix of g in bases $\mathcal{B}_3$ as input basis and $\mathcal{D}_4$ as output basis.

d. Determine the matrix of g in bases $\mathcal{D}_3$ as input basis and $\mathcal{D}_4$ as output basis.

Exercise 6 (4 points)

Let $h : \left\{ \begin{array}{lcl} \mathbb{R}^3 & \longrightarrow & \mathbb{R}^3 \\ (x,y,z) & \longmapsto & (2x+3y+z, x-z, x+y) \end{array} \right.$

a. Determine the matrix of M associated to h in the standard basis as input and output basis.

b. Let C_1, C_2, C_3 be the column vectors of this matrix. Is the family $\{C_1, C_2, C_3\}$ linearly independent? If it is not, find a maximal independent subfamily.

c. Deduce $\operatorname{Im} h$, $\operatorname{rank}(h)$ and $\operatorname{Ker} h$.

Exercise 7 (3 points)

Let $\mathcal{B}$ be the standard basis of $\mathbb{R}^3$ and $\mathcal{B}' = \{(1,3,-3);(6,2,-7);(1,0,-1)\}$ another basis of $\mathbb{R}^3$.

Let f be the linear map associated to the matrix $P = \begin{pmatrix} 1 & 6 & 1 \\ 3 & 2 & 0 \\ -3 & -7 & -1 \end{pmatrix}$ in the standard basis as input and output basis.

a. Without doing any computations, what can you say about the rank of f? How is the matrix P called?

b. Determine P^{-1} the inverse matrix of P, then check your final result.

Exercise 8 (4 points)

Let $f : \begin{cases} \mathbb{R}^3 & \longrightarrow & \mathbb{R}^3 \\ (x, y, z) & \longmapsto & \left(\frac{1}{2}(x-z), y, \frac{1}{2}(z-x)\right) \end{cases}$

a. Show that f is a projector.

b. Determine $\operatorname{Ker} f$ and $\operatorname{Im} f$ as spanned linear subspaces (using the Span notation), then find a basis $\mathcal{B}_1$ of $\operatorname{Ker} f$ and a basis $\mathcal{B}_2$ of $\operatorname{Im} f$.

c. We accept without proof that the union of the vectors from the two bases $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ is a basis of $\mathbb{R}^3$.
Find the matrix of f in basis $\mathcal{B}$ as input and output basis.