

EPITA

Mathematics

Final exam (S2)

May 2018

Name :

First Name :

Class :

MARK :

Exercise 1 (2 points)

Let $A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & -1 & 1 \\ 2 & 4 & 5 \end{pmatrix}$. Determine the inverse matrix A^{-1} . (Don't forget to check the final result on your draft.)

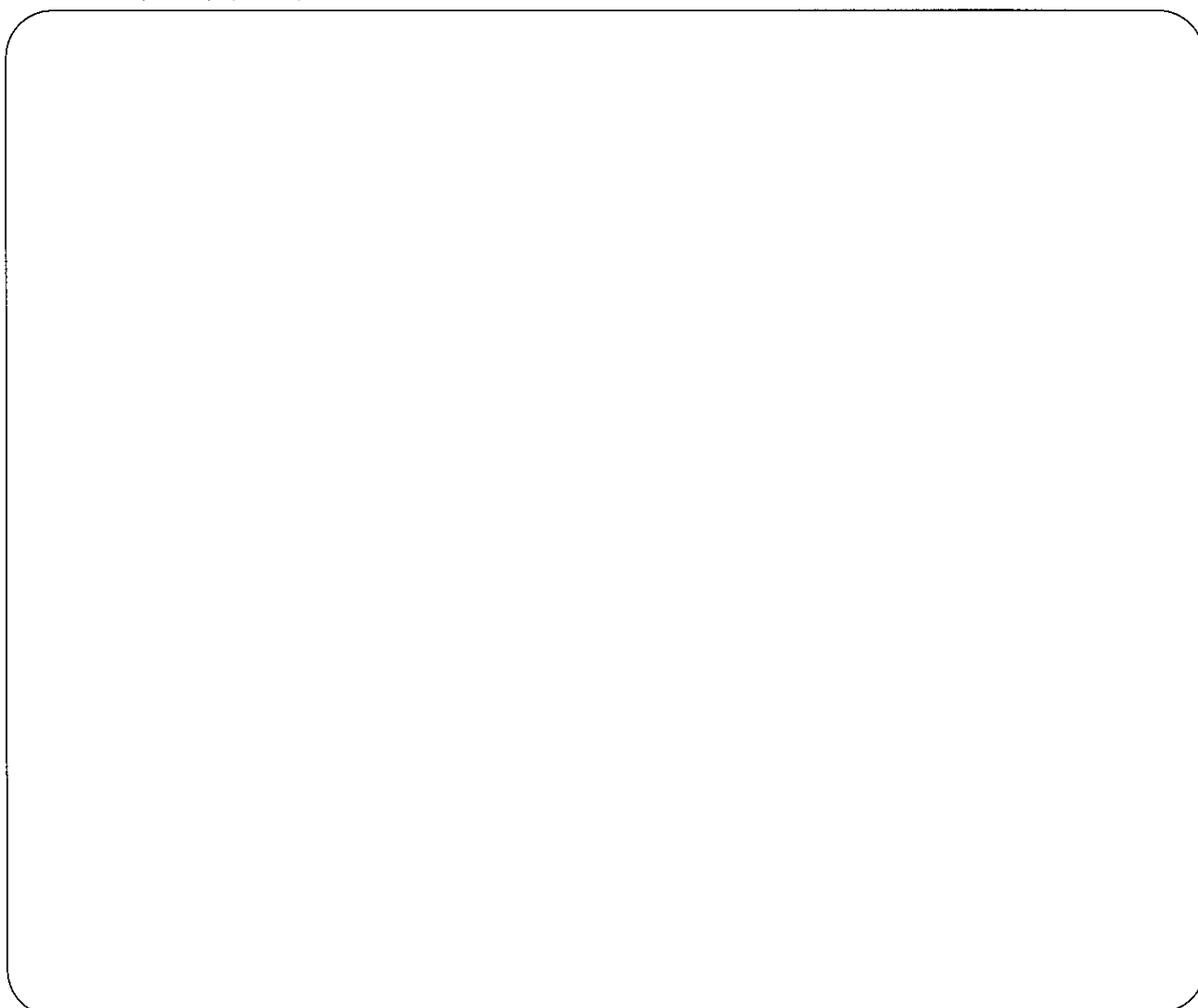
Exercise 2 (4 points)

Calculate the partial fractal decomposition in $\mathbb{R}(X)$ of the following rational fractions :

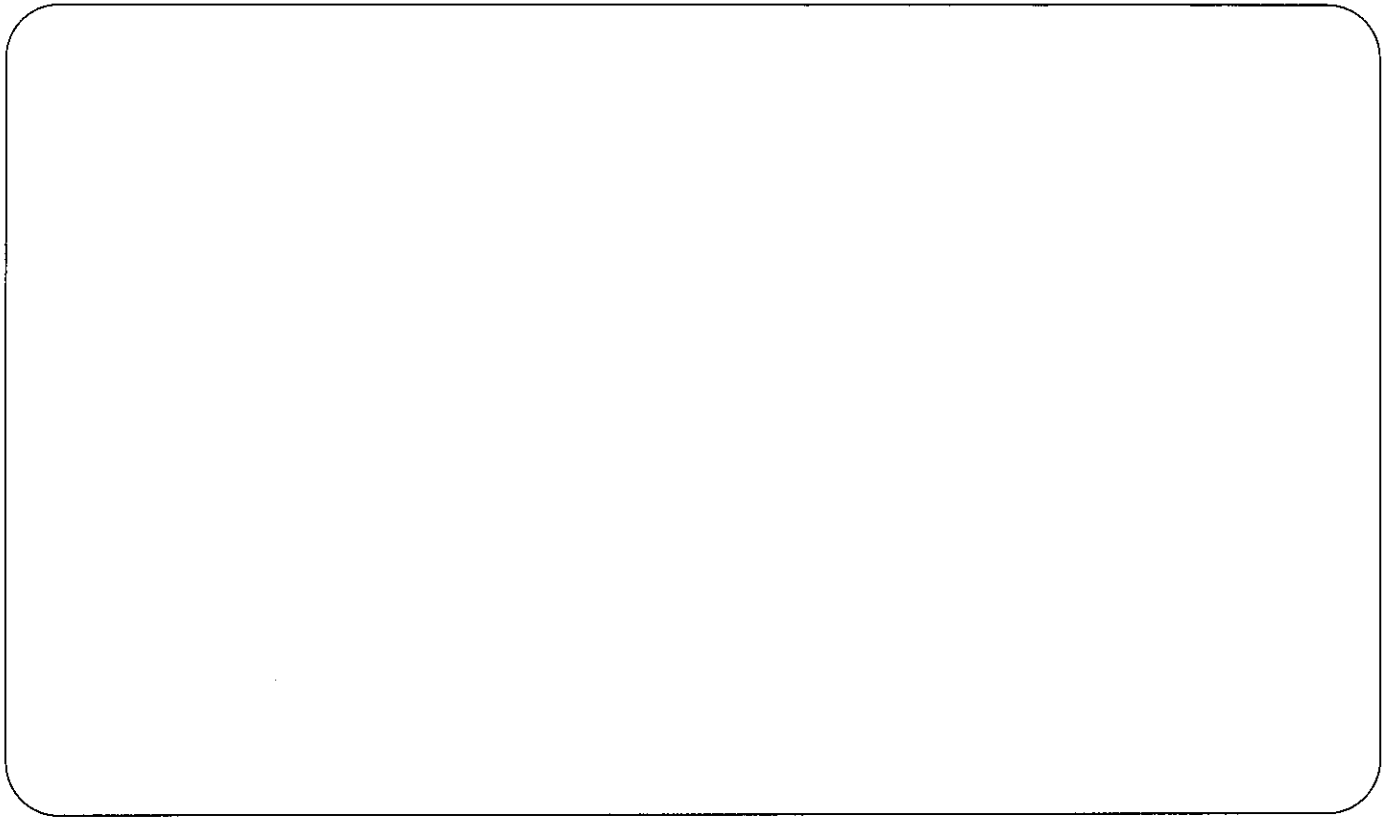
1. $F(X) = \frac{X^2 + X - 1}{(X - 1)(X - 2)(X + 2)}$



2. $G(X) = \frac{X^4 + X^3 + X^2 + 2X + 2}{(X - 1)^2(X + 1)}$

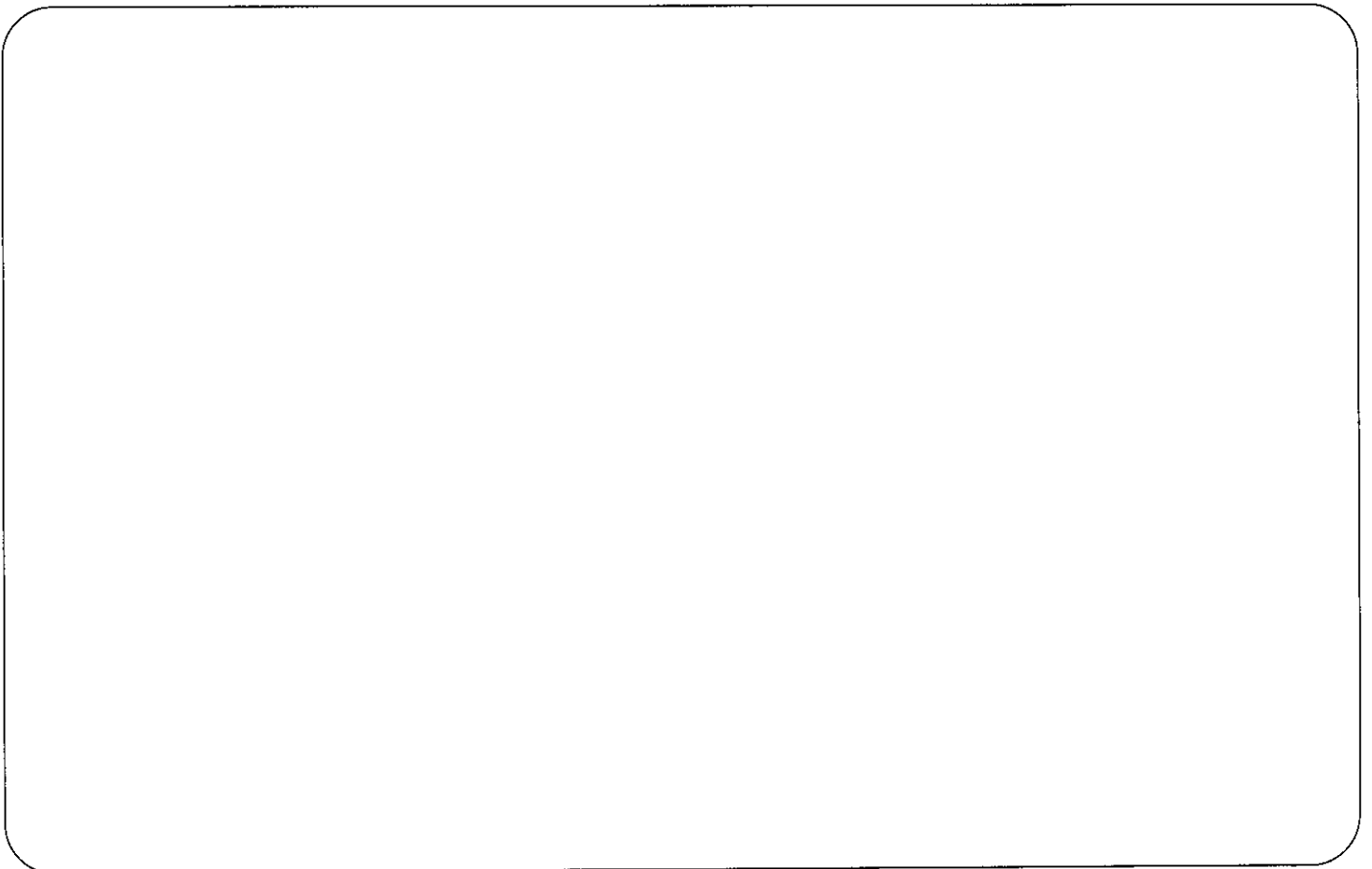


3. $H(X) = \frac{2X^2 - 1}{(X + 1)(X^2 + X + 1)}$



Exercise 3 (2 points)

Let E and F be two vector spaces over \mathbb{R} and $f \in \mathcal{L}(E, F)$. Prove that f is injective iff $\text{Ker}(f) = \{0\}$.



Exercise 4 (3 points)

1. Let $f : \mathbb{R}_2[X] \rightarrow \mathbb{R}_2[X]$ be defined for every $P \in \mathbb{R}_2[X]$ by $f(P(X)) = 2XP(X) - X^2P'(X)$.

Determine the matrix of f with respect to the standard basis of $\mathbb{R}_2[X]$.

2. Let $f \in \mathcal{L}(\mathcal{M}_2(\mathbb{R}))$ be defined by $f : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$.

Determine the matrix of f with respect to the standard basis of $\mathcal{M}_2(\mathbb{R})$.

Exercise 5 (2,5 points)

Let E be a vector space over \mathbb{R} and $u \in \mathcal{L}(E)$. We use the notation $u^2 = u \circ u$.
Show that $\text{Ker}(u) \cap \text{Im}(u) = \{0\} \iff \text{Ker}(u) = \text{Ker}(u^2)$

Exercise 6 (2,5 points)

You have to justify your answers to the two following questions.

1. Do the vectors $u = (1, 1, 0)$, $v = (4, 1, 4)$ and $w = (2, -1, 4)$ form a basis of \mathbb{R}^3 ?

2. Let $E = \mathbb{R}^{\mathbb{R}}$ be the vector space of functions from \mathbb{R} to \mathbb{R} , and let $(f, g) \in E^2$, defined for every $x \in \mathbb{R}$ by
- $$\begin{cases} f(x) = \frac{e^x + e^{-x}}{2} \\ g(x) = \frac{e^x - e^{-x}}{2} \end{cases}, \text{ and } F = \text{Span}(f, g). \text{ What is the dimension of } F?$$

Exercise 7 (5 points)

Let $f : \begin{cases} \mathbb{R}^3 & \rightarrow \mathbb{R}^3 \\ \begin{pmatrix} x \\ y \\ z \end{pmatrix} & \mapsto \begin{pmatrix} x + 2y \\ 3y \\ 2x - 4y + 2z \end{pmatrix} \end{cases}$ and let A be the matrix of f with respect to the standard basis \mathcal{B} of \mathbb{R}^3 .

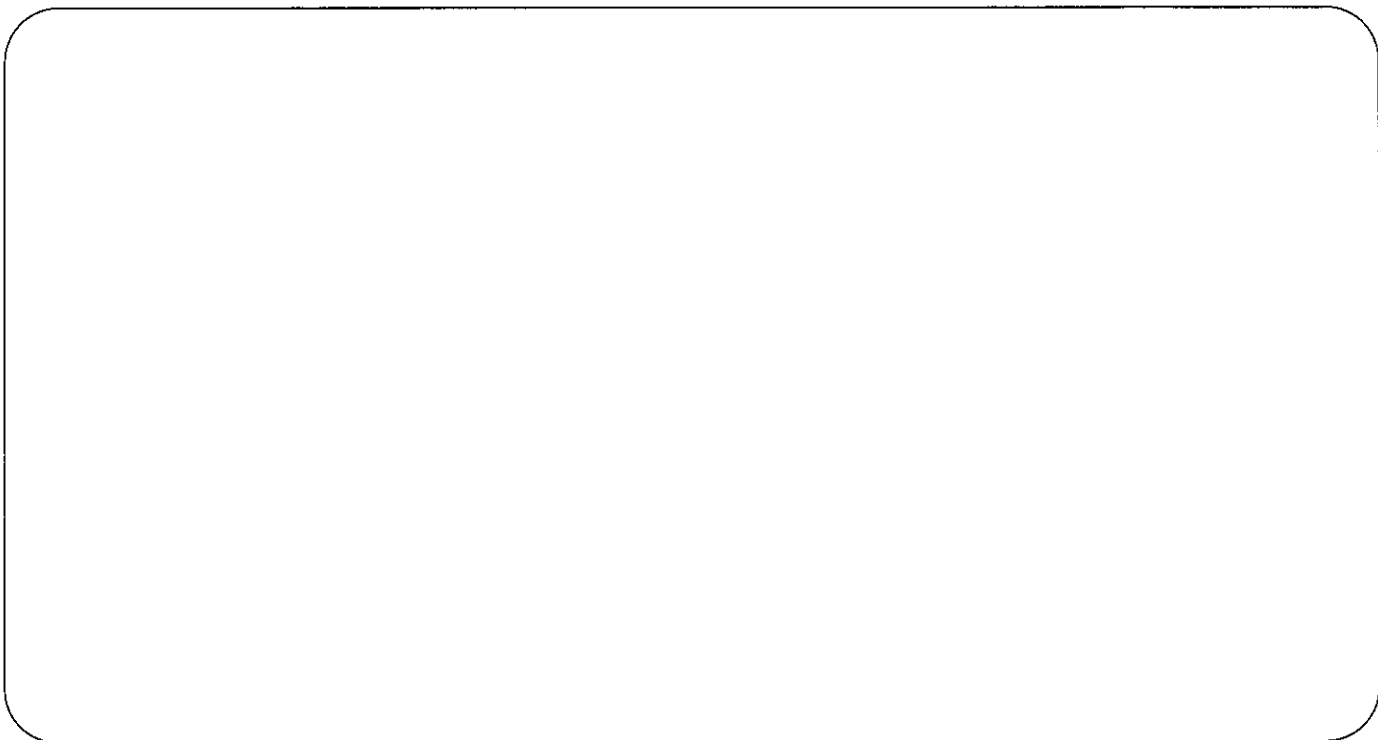
1. Determine A .

2. Let $u_1 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$, $u_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $u_3 = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$ so that $\mathcal{B}' = (u_1, u_2, u_3)$ is a basis of \mathbb{R}^3 .

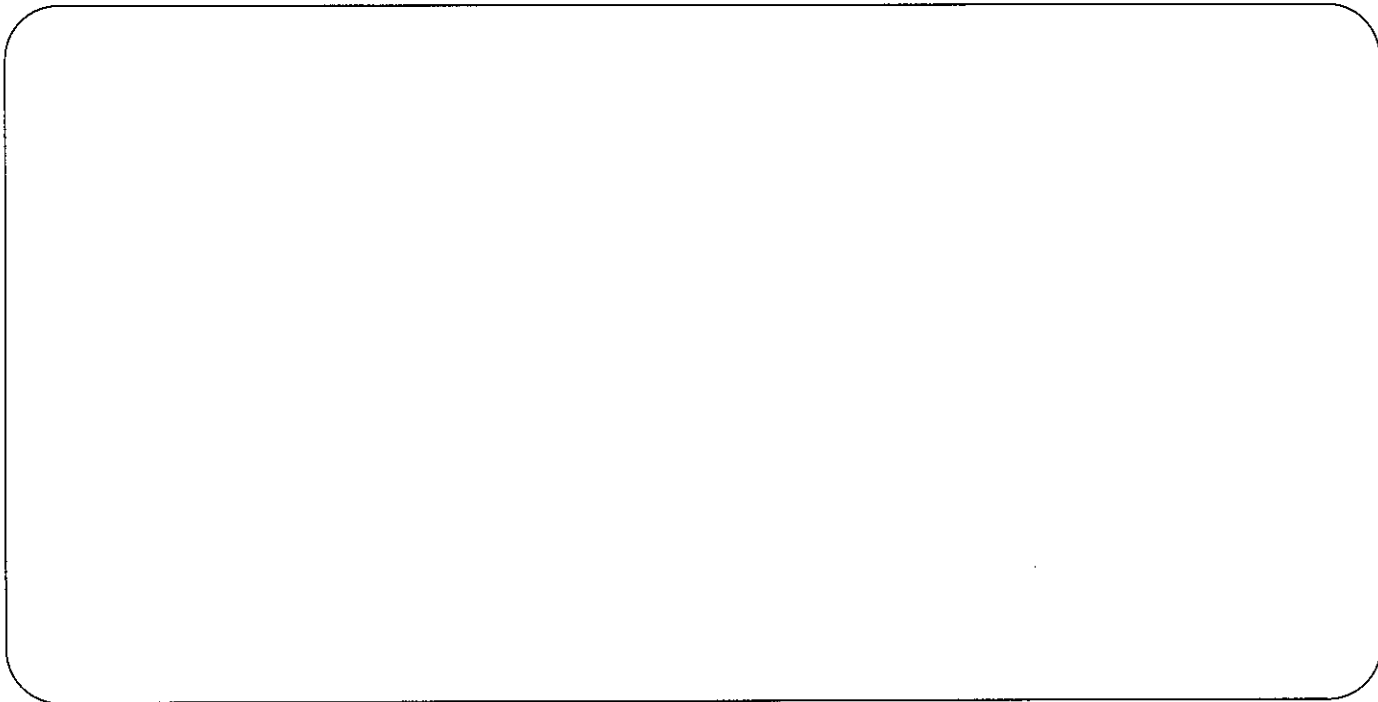
Determine the matrix of f with respect to \mathcal{B}' .

3. Let $P = \text{Mat}_{\mathcal{B}', \mathcal{B}}(id)$ where id is the identity map from \mathbb{R}^3 to \mathbb{R}^3 . Determine P^{-1} then $D = P^{-1}AP$. What can you remark?

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4. Calculate D^2 , D^3 and deduce (without induction) D^n for every $n \in \mathbb{N}^*$.



5. Deduce (without induction) A^n as a function of P , D and n for every $n \in \mathbb{N}^*$.

