EPITA

Mathematics

Final exam (S2)

May 2018

Name :

First Name :

Class :

MARK :

Exercise 1 (2 points)

Let $A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & -1 & 1 \\ 2 & 4 & 5 \end{pmatrix}$. Determine the inverse matrix A^{-1}. (Don't forget to check the final result on your draft.)

Exercise 2 (4 points)

Calculate the partial fractal decomposition in $\mathbb{R}(X)$ of the following rational fractions :

1. $F(X) = \dfrac{X^2 + X - 1}{(X-1)(X-2)(X+2)}$

2. $G(X) = \dfrac{X^4 + X^3 + X^2 + 2X + 2}{(X-1)^2(X+1)}$

3. $H(X) = \dfrac{2X^2 - 1}{(X+1)(X^2+X+1)}$

Exercise 3 (2 points)

Let E and F be two vector spaces over $\mathbb{R}$ and $f \in \mathscr{L}(E, F)$. Prove that f is injective iff $\mathrm{Ker}(f) = \{0\}$.

Exercise 4 (3 points)

1. Let $f : \mathbb{R}_2[X] \longrightarrow \mathbb{R}_2[X]$ be defined for every $P \in \mathbb{R}_2[X]$ by $f(P(X)) = 2XP(X) - X^2P'(X)$.

   Determine the matrix of f with respect to the standard basis of $\mathbb{R}_2[X]$.

2. Let $f \in \mathscr{L}(\mathscr{M}_2(\mathbb{R}))$ be defined by $f : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$.

   Determine the matrix of f with respect to the standard basis of $\mathscr{M}_2(\mathbb{R})$.

Exercise 5 (2,5 points)

Let E be a vector space over $\mathbb{R}$ and $u \in \mathscr{L}(E)$. We use the notation $u^2 = u \circ u$.
Show that $\text{Ker}(u) \cap \text{Im}(u) = \{0\} \iff \text{Ker}(u) = \text{Ker}(u^2)$

Exercise 6 (2,5 points)

You have to justify your answers to the two following questions.

1. Do the vectors $u = (1, 1, 0)$, $v = (4, 1, 4)$ and $w = (2, -1, 4)$ form a basis of $\mathbb{R}^3$?

2. Let $E = \mathbb{R}^{\mathbb{R}}$ be the vector space of functions from $\mathbb{R}$ to $\mathbb{R}$, and let $(f, g) \in E^2$, defined for every $x \in \mathbb{R}$ by $\begin{cases} f(x) = \dfrac{e^x + e^{-x}}{2} \\ g(x) = \dfrac{e^x - e^{-x}}{2} \end{cases}$, and $F = \text{Span}(f, g)$. What is the dimension of F ?

Exercise 7 (5 points)

Let $f : \left\{ \begin{array}{ccc} \mathbb{R}^3 & \longrightarrow & \mathbb{R}^3 \\ \begin{pmatrix} x \\ y \\ z \end{pmatrix} & \longmapsto & \begin{pmatrix} x+2y \\ 3y \\ 2x-4y+2z \end{pmatrix} \end{array} \right.$ and let A be the matrix of f with respect to the standard basis $\mathscr{B}$ of $\mathbb{R}^3$.

1. Determine A.

2. Let $u_1 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$, $u_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $u_3 = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$ so that $\mathscr{B}' = (u_1, u_2, u_3)$ is a basis of $\mathbb{R}^3$.

   Determine the matrix of f with respect to $\mathscr{B}'$.

3. Let $P = \text{Mat}_{\mathscr{B}',\mathscr{B}}(id)$ where id is the identity map from $\mathbb{R}^3$ to $\mathbb{R}^3$. Determine P^{-1} then $D = P^{-1}AP$. What can you remark?

[this frame continues on the next page]

4. Calculate D^2, D^3 and deduce (without induction) D^n for every $n \in \mathbb{N}^*$.

5. Deduce (without induction) A^n as a function of P, D and n for every $n \in \mathbb{N}^*$.