EPITA

Mathematics

Final exam (S2)

June, 2017

Name :

First name :

Class :

GRADE :

Exercise 1 (2 points)

Let $A = \begin{pmatrix} -1 & 3 & -1 \\ 3 & -5 & 2 \\ -1 & 2 & -1 \end{pmatrix}$. Determine the matrix A^{-1} (check the final result carefully on your draft).

Exercise 2 (4,5 points)

Expand into partial fractions of $\mathbb{R}(X)$ the following rational fractions :

1. $F(X) = \dfrac{X^2 + X + 1}{(X+1)(X-1)(X-3)}$

2. $G(X) = \dfrac{X^3 - X - 1}{(X+1)(X+3)}$

3. $H(X) = \dfrac{X^2 - X - 1}{(X-2)(X^2+1)}$

Exercise 3 (3 points)

1. Let $f : \mathbb{R}_2[X] \longrightarrow \mathbb{R}^2$ be the linear map defined for any $P \in \mathbb{R}_2[X]$ by $f\big(P(X)\big) = \big(P(1), P(2)\big)$.

   Determine the matrix of f in the standard bases of $\mathbb{R}_2[X]$ and $\mathbb{R}^2$.

2. Let $u : \begin{cases} \mathbb{R}^3 & \longrightarrow \mathscr{M}_2(\mathbb{R}) \\ (x,y,z) & \longmapsto \begin{pmatrix} x+y & y+z \\ x+z & 0 \end{pmatrix} \end{cases}$

   Determine the matrix of u in the standard bases of $\mathbb{R}^3$ and $\mathscr{M}_2(\mathbb{R})$.

Exercise 4 (3 points)

Let E be a vector space over $\mathbb{R}$ and $f \in \mathscr{L}(E)$. As usual, we denote $f^2 = f \circ f$.

1. Show that $\mathrm{Ker}(f) \subset \mathrm{Ker}(f^2)$.

2. Show that $\mathrm{Im}(f^2) \subset \mathrm{Im}(f)$.

3. Show that : $\mathrm{Im}(f) \cap \mathrm{Ker}(f) = \{0_E\} \Longleftrightarrow \mathrm{Ker}(f) = \mathrm{Ker}(f^2)$.

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Exercise 5 (2 points)

We are working in $\mathbb{R}_2[X]$. For each of the following questions, you must justify your answer.

1. Let $\mathscr{B}_1 = \{X^2 + X; X + 3\}$. Is it a spanning family of $\mathbb{R}_2[X]$?

2. Let $\mathscr{B}_2 = \{2; X + 1; 2X^2; X^2 + 3\}$. Is it a linearly independant family of $\mathbb{R}_2[X]$?

3. Let $\mathscr{B}_3 = \{1; X + 1; X^2 + 2X\}$. Is it a basis of $\mathbb{R}_2[X]$?

Exercise 6 (3 points)

Let a and b be two real numbers, $A = \begin{pmatrix} 1 & a & b \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix}$ and $J = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

1. Calculate J^2 and J^k for $k \geqslant 3$.

2. Express A as a function of I, J and J^2 where $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

3. Deduce A^n for any $n \in \mathbb{N}^*$.

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Exercise 7 (4 points)

Let $f : \begin{cases} \mathbb{R}^4 & \longrightarrow \mathbb{R}^3 \\ (x,y,z,t) & \longmapsto (x+y, 2x+y+z, x+t) \end{cases}$

1. Prove that f is a linear map.

2. Determine $\text{Ker}(f)$ and give its dimension.

3. Deduce $\text{Im}(f)$.