Correction of the exam S1B1 LE

Exercise 1: equation of degree two

1. *Solve in $\mathbb{C}$ the following equation: $2z^2 + 2\sqrt{3}z + 2 = 0$. Let z_1 and z_2 denote the two solutions.*

   The discriminant is $\Delta = -4$. Thus, $z_1 = \dfrac{-2\sqrt{3}+2i}{4} = -\dfrac{\sqrt{3}}{2} + \dfrac{1}{2}i$ and $z_2 = \overline{z_1} = -\dfrac{\sqrt{3}}{2} - \dfrac{1}{2}i$

2. *Find the exponential forms of z_1 and z_2.*

   $|z_1| = |z_2| = 1$. $z_1 = e^{\frac{5\pi}{6}i}$ and $z_1 = e^{\frac{7\pi}{6}i}$.

Exercise 2: logic

Consider a function $f : \mathbb{R} \longrightarrow \mathbb{R}$.

1. *Translate in mathematical language the following sentences (using quantifiers)*
   (a) "The equation $f(x) = 0$ admits at least one solution"

   $\exists\, x \in \mathbb{R},\ f(x) = 0$

   (b) "The function f is a constant function"

   $\exists\, a \in \mathbb{R},\ \forall\, x \in \mathbb{R},\ f(x) = a$

   (c) "The function f is upper bounded"

   $\exists\, M \in \mathbb{R},\ \forall\, x \in \mathbb{R},\ f(x) \leq M$

2. *Consider the following properties:*
   P: "$\forall\, x \in \mathbb{R},\ f(x) = 0$", Q: "$\exists\, x \in \mathbb{R},\ f(x) = 0$" *and* R: "$(\forall\, x \in \mathbb{R},\ f(x) > 0) \vee (\forall\, x \in \mathbb{R},\ f(x) < 0)$"

   (a) *Write the negation of P, of Q and of R.*

   $\neg(P) =$ "$\exists\, x \in \mathbb{R}, f(x) \neq 0$"

   $\neg(Q) =$ "$\forall\, x \in \mathbb{R}, f(x) \neq 0$"

   $\neg(R) =$ " $(\exists\, x \in \mathbb{R}, f(x) \leq 0) \wedge (\exists\, x \in \mathbb{R}, f(x) \geq 0)$ "

   (b) *Select in this table the implications which are true:*

$P \Longrightarrow Q$	$Q \Longrightarrow P$	$Q \Longrightarrow R$	$\neg(Q) \Longrightarrow \neg(P)$	$\neg(P) \Longrightarrow \neg(R)$
X			X	

Exercise 3: sets and functions

1. *Consider two sets E and F, a function $f : E \longrightarrow F$, $A \subset E$ and $B \subset F$. Write the mathematical definition of the sets $f(A)$ and $f^{-1}(B)$.*

$$f(A) = \{f(x); x \in A\} \text{ and } f^{-1}(B) = \{x \in E;\ f(x) \in B\}$$

2. *Using a figure, define a function $f : \{a, b, c, d\} \longrightarrow [\![1, 5]\!]$ which satisfies the three properties $f(\{a, b\}) = \{1, 2\}$, $f^{-1}(\{5\}) = \emptyset$ and $f^{-1}(\{2\}) = \{b, c\}$.*

   A possible answer, for example, is: $f(a) = 1$, $f(b) = 2$, $f(c) = 2$ and $f(d) = 4$

3. Let $g : \begin{cases} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & |x - 1| \end{cases}$

   (a) *Draw the graph of g.*

   We let you do it :)

(b) *Find $g(\{-1,2\})$, $g([-1,3])$, $g^{-1}(\{1\})$ and $g^{-1}([0,1])$.*

$g(\{-1,2\}) = \{2,1\}$, $g([-1,3]) = [0,2]$, $g^{-1}(\{1\}) = \{0,2\}$ and $g^{-1}([0,1]) = [0,2]$.

(c) *Is g injective? Justify. If not, find two intervals I_1 and J_1 such that $g : I_1 \longrightarrow J_1$ is injective.*

$g(0) = g(2) = 1$ and $1 \neq 2$. The function g is hence not injective. To make it injective, we can choose for example $I_1 = [1, +\infty[$ and $J_1 = \mathbb{R}$.

(d) *Is g surjective? Justify. If not, find two intervals I_2 and J_2 such that $g : I_2 \longrightarrow J_2$ is surjective.*

-2 has no pre-image, the function g is hence not surjective. To make it surjective, we can choose for example $I_2 = \mathbb{R}$ and $J_2 = \mathbb{R}^+$.

Exercise 4: relations

In $E = \mathbb{N}^*$, consider the relation $\mathcal{R}$ defined by: $\forall (a,b) \in E^2, a\,\mathcal{R}\,b \iff \exists n \in \mathbb{N}$ such that $b = a^n$.

1. *Is $\mathcal{R}$ reflexive? Justify.*

   Let $a \in E$. Then $a = a^1$ which implies that $a\mathcal{R}a$. $\mathcal{R}$ is hence reflexive.

2. *Is $\mathcal{R}$ symmetric? Justify.*

   $8 = 2^3$ which implies that $2\mathcal{R}8$. However, there is no $n \in \mathbb{N}$ such that $2 = 8^n$. Relation $\mathcal{R}$ is hence not symmetric.

3. *Is $\mathcal{R}$ transitive? Justify.*

   Let $(a,b,c) \in E^3$ such that $a\mathcal{R}b$ and $b\mathcal{R}c$. Then there exists $(n,p) \in \mathbb{N}^2$ such that $b = a^n$ and $c = b^p$.

   Thus, $c = a^{np}$ which leads to $a\mathcal{R}c$. The relation is transitive.

4. *Let $(a,b) \in E^2$ such that $a\mathcal{R}b$ and $b\mathcal{R}a$.*

   (a) *Show that there exists $(n,p) \in \mathbb{N}^2$ such that $b = b^{np}$.*

   Since $a\mathcal{R}b$ and $b\mathcal{R}a$, there exists $(n,p) \in \mathbb{N}^2$ such that $b = a^n$ and $a = b^p$. Thus, $b = (b^p)^n = b^{np}$.

   (b) *Deduce that $b = 1$ or $n = p = 1$. Finally, what have you proven in this question 4?*

   $b = b^{np} \implies b = 1$ or $np = 1$. In the case $np = 1$, since n and p are natural numbers, $n = p = 1$.

   If $b = 1$ then $a = b^p = 1$. If $b \neq 1$, then $b = a^n = a^1 = a$. In all cases, we get $a = b$. Relation $\mathcal{R}$ is hence antisymmetric.