CORRECTION OF EXAM S1 B1 DP 2023

Exercise 1: cardinal numbers

1. *Let F and G be two finite disjoint sets. What is Card$(F \cup G)$?*

$$\text{Card}(F \cup G) = \text{Card}(F) + \text{Card}(G)$$

2. *Let E be a finite set, A and B two subsets of E (maybe disjoint, maybe not).*

   (a) *Using a partition of A, find a relation between Card(A), Card$(A \setminus B)$ and Card$(A \cap B)$. You can draw the sets if it is helpful...*
   Remind that $A \setminus B = \{x \in E;\ x \in A \ \wedge \ x \notin B\}$.

   The set A can be expressed with the disjoint union $A = (A \setminus B) \sqcup (A \cap B)$. We can hence use the formula of question 1 with $F = A \setminus B$ and $G = A \cap B$. We get

$$\text{Card}(A) = \text{Card}\left((A \setminus B) \cup (A \cap B)\right) = \text{Card}(A \setminus B) + \text{Card}(A \cap B)$$

   (b) *Write a formula which enables one to get Card$(A \cup B)$.*

   $\text{Card}(A \cup B) = \text{Card}(A) + \text{Card}(B) - \text{Card}(A \cap B)$

   (c) *Using a partition of $A \cup B$ and question 2.(a), prove the formula of question 2.(b).*

   The set $A \cup B$ can be expressed as a disjoint union $A \cup B = (A \setminus B) \sqcup B$.
   Thus, $\text{Card}(A \cup B) = \text{Card}(A \setminus B) + \text{Card}(B) = \text{Card}(A) - \text{Card}(A \cap B) + \text{Card}(B)$ using question 2.(a).

Exercise 2: podiums

In a cycling race, there are 20 competitors. The podium will consist of the ranking (from 1 to 3) of the best three competitors. Peter and Jade are parts of the 20 competitors.

You will justify briefly your answers. *You will simplify your results, but do not compute the final numerical values.*

1. *How many podiums are possible?*

   For the winner of the race, there are 20 possibilities, for the 2^{nd} position, there are 19 possibilities remaining, and for the 3^{rd} position, 18 possibilities. Finally, $20 \times 19 \times 18$ podiums are possible.

2. *How many podiums are possible, such that Jade is ranked at first position?*

   Jade is ranked at first position, there are 19 possibilities remaining for the second position and 18 for the third position. There are 19×18 podiums such that Jade is at first position.

3. *How many podiums are possible, which contain neither Peter nor Jade?*

   If neither Jade nor Peter are on the podium, it remains 18 competitors who can be on the podium. We are hence in the same situation as in question 1, when replacing 20 with 18. The number of such podiums is $18 \times 17 \times 16$.

4. *How many podiums are possible, which contain Peter but do not contain Jade?*

   Jade is not on the podium, it remains 19 competitors (including Peter). Peter may be at 1^{st} position: in that case there are 18 possibilities for the 2^{nd} and 17 possibilities for the 3^{rd} position. Same computation if Peter is at position 2 or 3. Finally, there are $3 \times 18 \times 17$ podiums which contain Peter but do not contain Jade.

5. *How many podiums are possible, which contain at least Jade or Peter?*

   We count the complement set. There are $18 \times 17 \times 16$ podiums which contain neither Peter nor Jade. Thus, there are $20 \times 19 \times 18 - 18 \times 17 \times 16$ podiums containing at least one of them.

6. *The race organizers have decided to reward the best three competitors. There will be no distinction between them: each of them will be awarded a bicycle. How many ways of awarding the three bicycles are possible?*

   In this question, the order of the 3 competitors on the podium does not matter: each of them will be awarded a bicycle. We hence have to choose a 3-person subset from a 20-competitor set. The number of possibilities is $\binom{20}{3} = \frac{20!}{3! \times 17!} = 3 \times 19 \times 20$.

Exercise 3: counting

Let $(n, p) \in \mathbb{N}^2$.
A box contains p green balls and n red balls.

Let $k \in [\![0, n+p]\!]$ such that $k \leq n$ and $k \leq p$. We simultaneously pick k balls from the box.

1. *How many ways of picking the balls are possible?*

   There are $\binom{n+p}{k}$ ways of picking the balls.

2. *Let $i \in [\![0, k]\!]$. How many ways of picking (simultaneously) i green balls and $k-i$ red balls are possible?*

   There are $\binom{p}{i}$ ways of picking the i green balls and $\binom{n}{k-i}$ ways of picking the $k-i$ red balls. There are hence $\binom{p}{i} \times \binom{n}{k-i}$ ways of picking i green balls and $k-i$ red balls.

3. *Deduce the formula $\sum_{i=0}^{k} \binom{p}{i}\binom{n}{k-i} = \binom{n+p}{k}$. Justify accurately.*

   To pick k balls, you can pick: 0 green balls and k red balls, or 1 green ball and $k-1$ red balls, or 2 green balls and $k-2$ red balls, etc... until k green balls and 0 red balls. These different cases are disjoint. This leads to $\sum_{i=0}^{k} \binom{p}{i}\binom{n}{k-i}$ ways of picking k balls. However, we have shown at question 1 that there are $\binom{n+p}{k}$ ways of picking k balls. It results that:

$$\sum_{i=0}^{k} \binom{p}{i}\binom{n}{k-i} = \binom{n+p}{k}$$

4. *Application: compute $\sum_{i=0}^{4} \binom{4}{i}\binom{7}{4-i}$. Your final answer should be a numerical value (an integer number).*

   Here, $k = 4$, $p = 4$ and $n = 7$. Thus, $\sum_{i=0}^{4} \binom{4}{i}\binom{7}{4-i} = \binom{11}{4} = \frac{11!}{4!7!} = \frac{8 \times 9 \times 10 \times 11}{2 \times 3 \times 4} = 3 \times 10 \times 11 = 330$

Exercise 4: conditional probabilities

EPITA proposes voluntary maths support sessions, for S1 students requiring more explanations. This year, the probability that a student comes at the support sessions is 10%. If a student comes at the support sessions, his probability of validating the S1 Maths module is 80%. If a student does not come to the support sessions, his probability of validating is only 30%.
Consider the events V = "the student validates the S1 Maths Module" and S = "the student comes to the support sessions".
In the exercise, you will write your results as simplified fractions. They must be justified accurately.

1. *Translate the data of the exercise using probabilities and the events defined above.*

   The text of the exercise says that: $P(S) = 0,1$, $P(V \mid S) = 0,8$ and $P(V \mid \overline{S}) = 0,3$.

2. *We randomly pick an EPITA S1 student. What is the probability that he will validate the S1 Maths Module?*

   Since $\{S, \overline{S}\}$ is a partition of Ω, we know that $V = (V \cap S) \sqcup (V \cap \overline{S})$ (disjoint union). Therefore,

$$P(V) = P(V \cap S) + P(V \cap \overline{S}) = P(V \mid S)P(S) + P(V \mid \overline{S})P(\overline{S}) = 0,8 \times 0,1 + 0,3 \times 0,9 = 0,35 = \frac{7}{20}$$

3. *We randomly pick a student who has validated. What is the probability that he came to the support sessions?*

   We search the value of $P(S \mid V)$. We know that

$$P(S \mid V) = \frac{P(V \cap S)}{P(V)} = \frac{P(V \mid S)P(S)}{P(V)} = \frac{0,8 \times 0,1}{0,35} = \frac{0,08}{0,35} = \frac{8}{35}$$

4. *We want that a random student has a probability at least 75% of validating the S1 Maths Module. What should be the probability p of coming to the support sessions?*

   In that case, by reasoning as in question 1, we get $P(V) = 0,8p + 0,3(1-p) = 0,5p + 0,3$. We search values of p such that $0,5p + 0,3 \geq 0,75$. This leads to $p \geq 0,9$.

Exercise 5: lecture questions

Let X and Y be two finite random variables and $(a,b) \in \mathbb{R}^2$.

1. *Write the formulas which enable one to get $E(X+Y)$ and $E(aX+b)$.*
   See the lecture.
2. *Using the formulas of previous question (**highlight them** in your computations), show that Var$(X) = E(X^2) - (E(X))^2$.*
   See the lecture and the pdf file about proofs that you have to know.

Exercise 6: random variables

A student is doing an exam made of MCQ questions. The exam has 20 questions, each question counts for 1 point. The total mark at the exam is hence a mark out of 20. The questions have neither negative nor intermediate points: at each question, the mark is 0 or 1, no other values are possible.

The student did not prepare his exam and decides to answer randomly. His choices are independent and, at each question, he has a probability $p = \frac{1}{3}$ of choosing the right answer.

1. *Let $k \in [\![1,20]\!]$ and consider the random variable X_k = "Mark of the student at question k".*
   (a) *The value of $k \in [\![1,20]\!]$ being given, express the distribution of X_k.*

   The distribution is: $X_k(\Omega) = \{0,1\}$, $P(X_k = 1) = \dfrac{1}{3}$ and $P(X=0) = \dfrac{2}{3}$.
   The variables X_k are hence Bernoulli-distributed with the parameter $\dfrac{1}{3}$.

   (b) *Compute the expectation and the variance of X_k. Write the details of your computations.*

   - $E(X) = 0 \times P(X=0) + 1 \times P(X=1) = \dfrac{1}{3}$.
   - $E(X^2) = 0^2 \times P(X=0) + 1^2 \times P(X=1) = \dfrac{1}{3}$. Thus, $V(X) = E(X^2) - (E(X))^2 = \dfrac{1}{3} - \dfrac{1}{9} = \dfrac{2}{9}$

2. *Consider the random variable Y equal to the total mark of the student at the exam.*
   (a) *Express Y as a function of the variables X_k.*

   $Y = X_1 + X_2 + \cdots + X_{20}$ because Y is the sum of the points at each question. The X_k are independent.

   (b) *Find $P(Y=i)$ for all $i \in [\![0,20]\!]$. Justify your reasoning properly.*

   Since Y is the sum of 20 independent Bernoulli variables, Y is binomial-distributed with parameters 20 and $\dfrac{1}{3}$. Thus:

$$\forall i \in [\![0,20]\!],\ P(Y=i) = \binom{20}{i}\left(\frac{1}{3}\right)^i \left(\frac{2}{3}\right)^{20-i}$$

   (c) *Check that $\sum_{i=0}^{20} P(Y=i) = 1$.*
   Using binomial formula:

$$\sum_{i=0}^{20} P(Y=i) = \sum_{i=0}^{20} \binom{20}{i}\left(\frac{1}{3}\right)^i \left(\frac{2}{3}\right)^{20-i} = \left(\frac{1}{3} + \frac{2}{3}\right)^{20} = 1^{20} = 1$$

(d) *Compute the expectation and the variance of Y. Justify your computation.*

$E(Y) = E(X_1) + \cdots + E(X_{20}) = 20 \times \frac{1}{3}$ because the X_i have the same expectation.

Furthermore, since the $X_1, \cdots, X_{20}$ are independent, $V(Y) = V(X_1) + \cdots + V(X_{20}) = 20 \times \frac{2}{9} = \frac{40}{9}$.