EPITA

Mathematics

Midterm exam S1

Duration: 3 hours

November 2021

Name:

First name:

Class:

MARK:

The marking system is given for a grading scale from 0 to 30.
The final mark will be re-scaled from 0 to 20.

Instructions:

- Documents and pocket calculators are not allowed.
- Write your answers on the stapled sheets provided for answering. No other sheet will be corrected.
- Please, do not use lead pencils for answering.

Exercise 1 (3 points)

Consider the relation defined on $\mathbb{R}$ by:

$$\forall (x, y) \in \mathbb{R}^2,\ x \mathcal{R} y \iff x^2 \leq y^2$$

Enumerate the different properties (with the quantifiers) which define an order relation. For each property, say whether it is satisfied or not by the relation defined above. Justify your answer.

Exercise 2 (3 points)

Express in mathematical language, with the quantifiers, the following properties.

1. Every positive real number is the square of a real number.

2. If the product of two real numbers is zero, then at least one of these two numbers is zero.

3. Every natural number is even or odd. (Using the congruence relation is not allowed)

Exercise 3 (4,5 points)

1. Using a primitive (anti-derivative), compute $I = \displaystyle\int_0^1 (x^2+1)\sqrt{x^3+3x+1}\,\mathrm{d}x$

2. Using an integration by parts, compute $J = \displaystyle\int_0^1 (2x+1)e^{2x}\,\mathrm{d}x$

3. Using the substitution $x = 2t+1$, compute $K = \displaystyle\int_1^{2\sqrt{3}+1} \frac{1}{(x-1)^2+4}\,\mathrm{d}x$

Exercise 4 (3,5 points)

Consider the function $f : \mathbb{N} \longrightarrow \mathbb{N}$ defined by $f(n) = \left| \begin{array}{ll} \frac{n}{2} & \text{if } n \text{ is even} \\ n & \text{otherwise} \end{array} \right.$

1. Is f injective? Justify your answer.

2. Is f surjective? Justify your answer.

3. Assume in this question that the input set of f is $E = \{0, 1, 2, 3, 4, 5, 6\}$ (instead of $\mathbb{N}$).
   Find $f(\{0, 1, 2, 3\})$, $f^{-1}(\{1, 3\})$ and $f^{-1}(\{4\})$.

Exercise 5 (5 points)

In a box, there are n balls labeled 1 through n.
Let $k \in [\![1, n]\!]$. Assume that you **simultaneously** take k balls from the box.

1. How many possible results? Explain briefly your answer.

2. How many results contain the ball labeled 1? Explain briefly.

3. How many results do not contain the ball labeled 1? Explain briefly.

4. Explain why the previous results enable one to prove the following formula (P):

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

5. Prove this formula (P) with a computation, using the literal expression of the binomial coefficients, with factorials.

Exercise 6 (5 points)

Consider a 6-side fair dice.

1. Imagine a simple random experiment with the dice, which leads to a random variable X such that X is Bernoulli-distributed with the parameter $p = \frac{1}{6}$. What is the expectation and the variance of X?

2. Imagine a simple random experiment with the dice, which leads to a random variable Y such that Y is binomial-distributed with the parameters $n = 10$ and $p = \frac{1}{6}$. Express the expectation and the variance of Y as functions of n and p.

3. In this question, we roll the dice only once. The game is the following: we score 5 points if the dice result is a multiple of 3, 3 points if the dice result is 1 or 2, zero points otherwise.
   Let S be the score. Give the distribution of S. Then find its expectation and its variance.

Exercise 7 (4 points)

Imagine that a virus affects the world's population. A screening test has been developed. Its performances are the following:

- If a patient is not contaminated, the test gives the right answer with the probability 80%.
- If a patient is contaminated, the test gives the right answer with the probability 99%.

Assume that 25% out of the population is contaminated. We randomly pick a person and do the test.

Consider the events C: "This person is contaminated" and T: "This person is tested positive".

1. Give a partition of the event T in term of C. Write the literal formula which enables one to get $P(T)$. Specify the numerical value of each probability in this formula (the final numerical value of $P(T)$ is not required).

2. Explain how to compute the probability that this person is contaminated given that she is tested positive (the final numerical value is not required).

Exercise 8 (2 points)

Let $q \in \mathbb{R}$. For any $n \in \mathbb{N}$, consider the sum

$$S_n = \sum_{k=0}^{n} q^k = q^0 + q^1 + q^2 + \ldots + q^n$$

1. What is the value of S_n when $q = 1$?

2. Assume that $q \neq 1$. Show by induction that, for every $n \in \mathbb{N}$, $S_n = \dfrac{1 - q^{n+1}}{1 - q}$.