Brief correction of midterm n°1 in Physics

Exercise I

Part A

1) You had to derive with respect to time the functions $\begin{cases} x(t) = A(\omega t - \sin(\omega t)) \\ y(t) = A(1-\cos(\omega t)) \end{cases}$

Just few comments about notations for velocity and acceleration components. $\vec{v}$ is a vector which in Cartesian basis can be written as $\vec{v} = v_x\, \overrightarrow{u_x} + v_y\, \overrightarrow{u_y}$ where (v_x, v_y) are the components.

Another notation is the system $\begin{cases} v_x = \dot{x}(t) = A\omega(1-\cos(\omega t)) \\ v_y = \dot{y}(t) = A\omega\sin(\omega t) \end{cases}$. Note that there no arrow on vector components!

The acceleration reads $\begin{cases} a_x = \ddot{x}(t) = A\omega^2\sin(\omega t) \\ a_y = \ddot{y}(t) = A\omega^2\cos(\omega t) \end{cases}$.

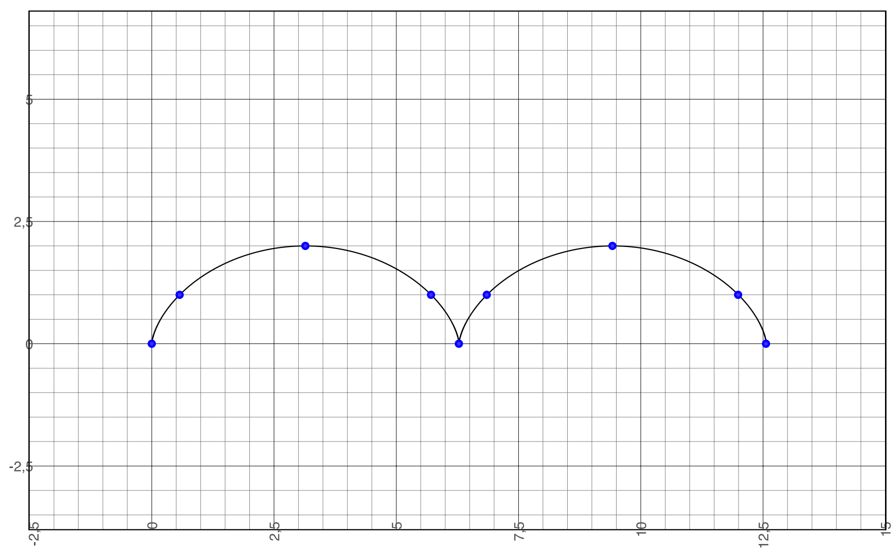
2) The norm of velocity vector reads $||\vec{v}|| = \sqrt{v_x^2 + v_y^2} = \sqrt{A^2\omega^2(1-\cos(\omega t))^2 + A^2\omega^2\sin^2(\omega t)}$

Then after simplification $||\vec{v}|| = A\omega\sqrt{1 - 2\cos(\omega t) + \cos^2(\omega t) + \sin^2(\omega t)} = A\omega\sqrt{4\sin^2(\frac{\omega t}{2})}$

$$||\vec{v}|| = \mathbf{2A\omega\sin(\frac{\omega t}{2})}$$

For acceleration, it's easier $||\vec{a}|| = \mathbf{A\omega^2}$**.**

3) Here pay attention on variables. We were looking for graph $(y = f(x))$, the so-called cycloid. Just brief notation: $t_k = k.\frac{T}{4} = \frac{2\pi k}{\omega}\frac{1}{4}$ and for the question $k \in [\![0,8]\!]$.



Part B

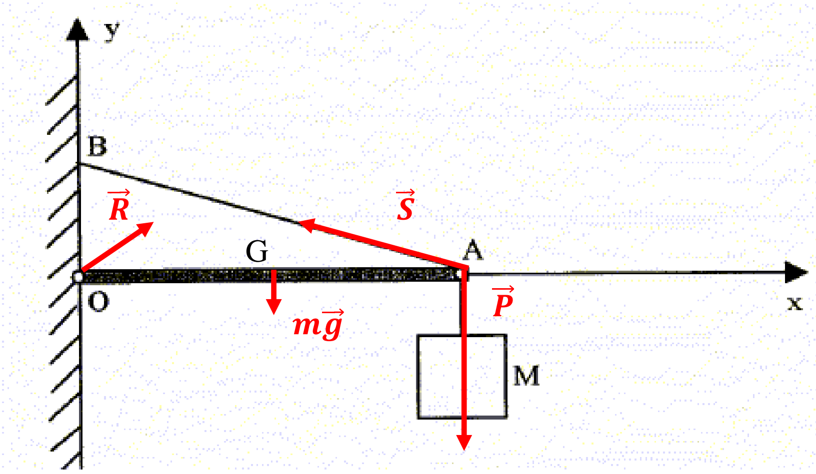
1) We were studying the polar coordinates $\begin{cases}\rho(t) = \rho_O e^{\omega t} \\ \theta(t) = \omega t\end{cases}$. The velocity vector can be written as $\vec{v} = \dot{\rho}\overrightarrow{u_\rho} + \rho\dot{\theta}\overrightarrow{u_\theta} = \rho_O \omega e^{\omega t}\overrightarrow{u_\rho} + \rho_O \omega e^{\omega t}\overrightarrow{u_\theta} = \boldsymbol{\rho(t)\omega(\overrightarrow{u_\rho} + \overrightarrow{u_\theta})}$.

2) By definition the norm of $\vec{v}$ is $\|\vec{v}\| = \sqrt{v_\rho^2 + v_\theta^2} = \rho(t)\omega\|\overrightarrow{u_\rho} + \overrightarrow{u_\theta}\| = \boldsymbol{\rho(t)\omega\sqrt{2} = v}$.

3) We recall that $\vec{v} = v\,\overrightarrow{u_T} = R(t)\dot{\theta}\,\overrightarrow{u_T} = R(t)\omega\,\overrightarrow{u_T}$ where $(\overrightarrow{u_T}, \overrightarrow{u_N})$ is the Frenet's basis.
Please here do NOT divide by unitary vector $\overrightarrow{u_T}$! It doesn't exist, no meaning!
Then one deduces $\boldsymbol{R(t)} = \frac{\boldsymbol{v}}{\boldsymbol{\omega}} = \boldsymbol{\rho(t)\sqrt{2}}$.

4) In Frenet's basis acceleration components reads $\begin{cases}\boldsymbol{a_T} = \frac{\boldsymbol{dv}}{\boldsymbol{dt}} = \boldsymbol{\rho(t)\omega^2\sqrt{2}} \\ \boldsymbol{a_N} = \frac{\boldsymbol{v^2}}{\boldsymbol{R}} = \boldsymbol{\rho(t)\omega^2\sqrt{2}}\end{cases}$ by using expression of v found in question 2.

<u>**Exercise II**</u>



1) First we define the studied system, namely here the beam. Four forces are acting on beam:

1° wall reaction $\vec{R}$ at point O
2° weight of beam $m\vec{g}$ acting at center-of-mass G
3° weight $\vec{P}$ of mass M acting at point A
4° cable stress $\vec{S}$ pointing from A to B

2) a) The rotation equilibrium with respect to O reads

$$\vec{0} = \sum \overrightarrow{\mathcal{M}_O}(\vec{F}_{ext}) = \overrightarrow{\mathcal{M}_O}(\vec{P}) + \overrightarrow{\mathcal{M}_O}(\vec{S}) + \overrightarrow{\mathcal{M}_O}(\vec{R}) + \overrightarrow{\mathcal{M}_O}(m\vec{g})$$

By definition $\overrightarrow{\mathcal{M}_O}(\vec{R}) = \overrightarrow{OO} \wedge \vec{R} = \vec{0}$ and using same formula one gets

$$\vec{0} = OG.mg.\sin\left(-\frac{\pi}{2}\right)\overrightarrow{u_z} + OA.Mg.\sin\left(-\frac{\pi}{2}\right)\overrightarrow{u_z} + OA.S.\sin\left(\pi - \frac{\pi}{6}\right)\overrightarrow{u_z}$$

Beware that this is a vector equality! After projection on $\overrightarrow{u_z}$ and simplification $(OA = L)$

$$0 = -\frac{L}{2}.mg - MgL + L.S.\sin\left(\frac{\pi}{6}\right)$$

$$\boldsymbol{S = mg + 2Mg = 3400\ N}$$

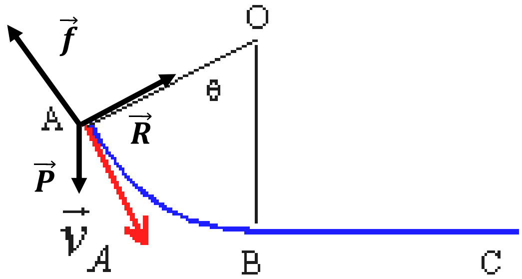
b) The translation equilibrium is the usual Newton's second law $\vec{0} = \sum \vec{F}_{ext}$.

After projecting on x- and y-axes

$$\begin{cases} 0 = R_x - S.\cos(\frac{\pi}{6}) \\ 0 = R_y + S.\sin\left(\frac{\pi}{6}\right) - mg - Mg \end{cases} \Leftrightarrow \begin{cases} \boldsymbol{R_x = S.\frac{\sqrt{3}}{2} = 1700\sqrt{3}\ N} \\ \boldsymbol{R_y = -\frac{S}{2} + mg + Mg = 200\ N} \end{cases}$$

c) Finally the norm is equal to $\|\overrightarrow{\boldsymbol{R_{wall}}}\| = \sqrt{\boldsymbol{R_x^2 + R_y^2}} = \sqrt{\boldsymbol{8710000}}\ \boldsymbol{N}$

Exercise III



1) a) Please do not consider the velocity vector $\overrightarrow{v_A}$ sketch above as a force! The only forces are the weight $\vec{P}$, the ground reaction $\vec{R}$ and the friction force $\vec{f}$.

b) The kinetic energy theorem reads for path AB $\boldsymbol{\Delta_{A\to B}\ E_k = \mathcal{W}_{A\to B}\left(\vec{R} + \vec{P} + \vec{f}\right)\ (E)}$.
Then using definition of work: $\mathcal{W}_{A\to B}\left(\vec{R} + \vec{P} + \vec{f}\right) = \int_A^B (\vec{R} + \vec{P} + \vec{f}).\overrightarrow{dl}$ where $\overrightarrow{dl}$ is an infinitesimal shift along the trajectory.
This means that $\overrightarrow{dl}$ is a length element tangent to the local point where we are integrating. Between A and B, the mass has a circular motion which can be described by polar coordinates and basis $(\overrightarrow{u_\rho}, \overrightarrow{u_\theta})$, thus $\overrightarrow{dl} = Rd\theta\overrightarrow{u_\theta}$ here.
What's more $\vec{R}$ is orthogonal to motion (we consider this force as the normal ground reaction, the total reaction is given in last question) which implies that $\vec{R}.\overrightarrow{dl} = 0$ at any point of the trajectory.
For friction force $\vec{f}$ we have claimed that $\vec{f}$ is collinear to $\vec{v} = R\dot{\omega}\overrightarrow{u_\theta}$ but with opposite orientation. So $\vec{f}.\overrightarrow{dl} = -f.R.d\theta$ and $\int_A^B d\theta = \theta_B - \theta_A = \theta$.
Last force $\vec{P}$: conservative force so use the property $\mathcal{W}_{A\to B}(\vec{P}) = -\Delta_{A\to B}\ E_p(\vec{P}) = mg(z_A - z_B)$
Finally, the first equation ***(E)*** reads $\frac{1}{2}m(v_B^2 - v_A^2) = mg(z_A - z_B) - fR\theta$
Reference frame is chosen such that $z_B = 0$.
So $\mathbf{f = \frac{1}{R\theta}\left(mgR(1 - \cos(\theta)) - \frac{1}{2}m\left(v_B^2 - v_A^2\right)\right) = \frac{1}{\frac{3}{2}}\left(\left(1 - \frac{1}{2}\right).\frac{3}{2} - \frac{1}{2}.0,1.(4 - 9)\right) = 0,34\ N}$

2) a) Between B and C weight is not working because z is constant. Kinetic energy theorem reads now $\Delta_{B\to C}\ E_k = \mathcal{W}_{B\to C}(\vec{f})$. Please read question before solving any exercise! Here frictions $\vec{f}$ are known and $\overrightarrow{v_C}$ must be expressed as function of f.
Writing explicitly the previous equation gives the answer: $\boldsymbol{\frac{1}{2}m\left(v_C^2 - v_B^2\right) = -f.BC}$ and numerically $\boldsymbol{v_C = \sqrt{5}\ m.s^{-1}}$

b) The total reaction $\overrightarrow{R_{tot}}$ takes both normal reaction $\vec{R}$ and frictions $\vec{f}$ into account: $\boldsymbol{\overrightarrow{R_{tot}} = \vec{R} + \vec{f}}$.
These two forces are orthogonal so the norm is simply given by $\|\overrightarrow{\boldsymbol{R_{tot}}}\| = \sqrt{\boldsymbol{R^2 + f^2}} = \sqrt{\boldsymbol{1,01}} \approx \boldsymbol{1\ N}$